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OPTIMIZATION OF A CELESTIAL-INERTIAL NAVIGATION SYSTEM USING SINGLE-BODY TRACKING

by

David Thomas Friest

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I. INTRODUCTION

Inertial navigation may be defined as the process of navigating a vehicle based on measurements made entirely internally in accordance with the Newtonian laws of motion and gravitation. Such a navigation system must therefore encompass devices for measuring acceleration, establishing a known coordinate system in which these accelerations are measured, and performing the necessary mathematical computations. These devices generally take the form of accelerometers, gyroscopes and an analog or digital computer. The accelerometers and gyroscopes are commonly mounted on a platform which is attached to the vehicle by means of a set of gimbals, the gimbals serving to isolate the instruments from vehicle rotation.

Accelerometers presently in use take many different forms. One of the simplest conceptually would be a spring-mass combination and a pick-off to detect deflection of the mass from its neutral position. From knowledge of the spring constant and proof-mass magnitude and deflection, non-gravitational acceleration can be computed. The fact that gravitational acceleration cannot be detected is quite important in the mechanization of inertial navigation systems and in analysis of error propagation. This fact can be seen quite clearly by analyzing the spring-mass accelerometer shown in Figure 1 with its sensitive axis in the vertical direction. The total acceleration of the case is $\ddot{R}$ upward, relative to the center of the earth. Summation of forces acting on the proof-mass m yields $Kx + D\dot{x} - mg = m(\ddot{R} - \ddot{x})$, or $\ddot{R} + g = \frac{1}{m}(Kx + D\dot{x} + m\ddot{x})$, where g is the acceleration of gravity, D is the damping constant and K is the spring constant. It is apparent from the above equation that the acceleration

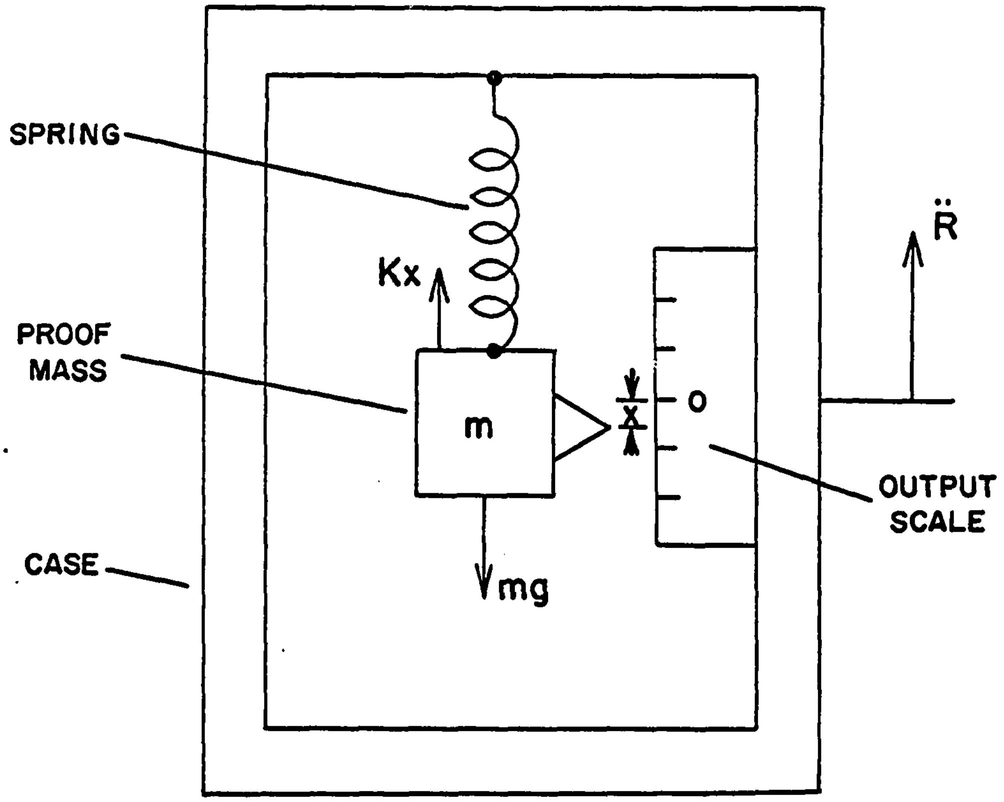

Figure 1. Ideal spring-mass accelerometer

$\ddot{R} + g$ is what is sensed. In steady-state in fact, the accelerometer output, x, is proportional to $\ddot{R} + g$. Since the total case acceleration is $\ddot{R}$ upward, the non-gravitational acceleration must be $\ddot{R} + g$ upward, since the gravitational acceleration is g downward. Thus, only non-gravitational accelerations are detected by an accelerometer. To determine total vehicle acceleration, it is therefore necessary to compute gravitational acceleration. This can be done from a knowlege of vehicle position, which can in turn be obtained from the inertial system. It is apparent that such an operation leads to feedback loops within the system; these result in a unique form of error propagation, as will be seen later.

Gyroscopes (or gyros) utilize the principle that the angular momentum vector of a rotating body, in the absence of disturbing torques, remains fixed relative to inertial space; and, if torques are present, the time rate of change of angular momentum with respect to inertial space is equal to the applied torque. Gyros conventionally consist of a rotor of substantial moment of inertia spinning very rapidly about the axis of symmetry, mounted within a case that allows the rotor three degrees of freedom relative to the platform. Pick-off devices are used to detect rotation of the rotor relative to the platform about one or both of the axes orthogonal to the angular momentum vector. Signals from these pick-offs indicate rotation of the platform relative to inertial space, and such signals are suitably shaped, amplified and applied to torquing devices mounted in the gimbal system. In the ideal situation, then, a known platform orientation is maintained in which acceleration is measured. If it is desired to rotate the platform relative to inertial space, torques of known magnitude and time duration can be applied to the gyros, resulting in known changes in direction of the angular momentum vectors. The platform torques or servos then will cause the platform to rotate at a known rate, thus maintaining a rotating platform coordinate system.

The analog or digital computer present in an inertial navigation system is used primarily to solve the Newtonian equations of motion and to compute gravitational acceleration, based on the computed position of the vehicle. The computer is also used to compute gyro torquing rates, if such are required, and to present computed position and velocity of the vehicle to personnel on board or an automatic pilot, so that vehicle steering may be accomplished. At a minimum, then, the computer is required to possess facilities for accepting inputs from the accelerometers and generating outputs to the gyros, an integration capability including an accurate clock, and algebraic capabilities.

The primary advantages of an inertial (or pure inertial) navigation system, particularly for military applications, are an immunity to cloud cover or other bad weather and the ability to operate without radiation of any form emanating from the vehicle. The disadvantages of such a system are the requirement of initial position and velocity of the vehicle and the fact that system position errors tend to grow without bound, as will be seen later.

Since the development of the first pure inertial navigation system about 1946, many varieties of hybrid or combined systems have been developed which tend to minimize these disadvantages. One of the simplest systems (actually a system operating technique) consists of determining vehicle position from known landmarks or celestial observations (when possible) and periodically correcting the inertial system position indication.

Another more complex combined system utilizes an independent source of vehicle velocity; for example, a Doppler radar or air-speed indicator in an aircraft, or a water speed indicator in a ship. This velocity signal can be suitably combined with inertial system signals, resulting in reduced system error, although errors still tend to grow without bound. Such a system is commonly known as a damped-inertial system, for reasons which will be clear later.

Another combined system is the stellar-inertial or celestial-inertial system. This system utilizes a star-tracker, generally mounted on the platform with two degrees of freedom, and a suitable window. Determination of the lines-of-sight to two stars establishes an inertially fixed coordinate system. Thus platform orientation can be monitored and corrected. Such a system results in much smaller system errors than those of a pure inertial system at a given time, although errors still grow without bound. This system, of course, requires some clear sky for operation, which is a disadvantage.

A damped-stellar-inertial system is the logical combination of both of the above systems, and it will be seen that such a system results in bounded errors. This system, however, retains the disadvantage of requiring some clear sky.

Another system which has been proposed recently for marine applications is a modification of the damped-stellar-inertial system which utilizes an external source of reference velocity and a sun-tracker. The sun-tracker would operate in that portion of the spectrum which is not appreciably attenuated by cloud cover, thus giving an all-weather capability. One apparent disadvantage of such a system is that only one line-of-sight is obtained; thus platform misalignment about the line-of-sight can not be instantaneously detected. It will be seen, however, that this limitation and another limitation can be successfully circumvented, and that the performance of such a system can be expected to be quite satisfactory.

It is the object of this dissertation to present an optimum mechanization scheme for such a damped-celestial-inertial navigation system. By optimization is meant the realization of a system mechanization resulting in the least possible mean-square system error, under the constraints presented by the capacity and capabilities of a typical, special-purpose digital computer, such as is used currently in marine systems.

Before the optimum mechanization equations are derived, it is necessary to present basic inertial system mechanization and error propagation equations. These are given in Sections II and III. Section IV presents a derivation of the basic information available from a single line-of-sight, discusses the fundamental limitation present, and shows a method of circumventing this limitation. In Section V the general equations which give the absolute optimum mechanization are derived, and several methods for solving these equations are discussed. These equations are based on "open-loop" system operations; that is, during observation and filtering of line-of-sight data, it is assumed that no corrections are applied. Section VI presents a method of "closing the loops", or observing and filtering line-of-sight data while applying corrections in an optimal manner.

Sections VII and VIII present, respectively, a computationally practical method of solution of the equations derived in Section V and a numerical example showing system performance. Certain numerical quantities used in Section VIII are not intended to reflect the present state-of-the-art in inertial navigation, as such performance figures are classified. These quantities, primarily random gyro drift rate variance, are meant to be regarded only as "normalized" or "postulated" values, used for the sake of obtaining a measure of system performance under such conditions.

The material presented in Sections I through III is discussed in much more detail in Pitman (1) and Pinson (2), which form the basis of these sections. The reader is referred to these two sources, particularly Pitman which is more readily accessible, for excellent and thorough discussions of this material.

II. INERTIAL NAVIGATION SYSTEM MECHANIZATION EQUATIONS

There are many possible forms which inertial navigation system mechanization equations may take. These forms depend primarily on the coordinate system chosen for use with the system. Since a marine system is being studied here, a "natural" coordinate system is the so-called longitude-latitude system, in which the inertial platform is torqued to maintain it locally level with one axis of the platform coordinate system always pointed north. With such a system, two single-axis accelerometers and three single-axis gyros are required. The sensitive axes of the accelerometers will be assumed oriented in a locally-level plane and pointing east and north. The gyro sensitive axes will be assumed oriented east, north and up (vertically), thus allowing complete control of the platform.

The fundamental problem to be solved is the determination of vehicle position and velocity relative to the earth, based on non-gravitational acceleration measurements made in a coordinate system rotating with respect to another coordinate system fixed in inertial space, and the determination of platform torquing rates required to maintain the desired platform orientation.

First, a set of inertially fixed axes is required. This set is chosen as one non-rotating with respect to the stars, with its origin fixed at the center of the earth. Such a choice ignores both the gravitational attraction of the sun, moon, planets and stars and the corresponding acceleration of the earth. On or near the surface of the earth, however, these two effects so very nearly cancel that the above choice is quite adequate, as indicated by Pitman (1) and Pinson (2). This coordinate system will be denoted by the subscript "I".

Another coordinate system required is one with its origin at the center of the earth and fixed with respect to the earth, denoted by the subscript "E". It rotates with respect to coordinate system "I" at a vector angular velocity of $\underline{\Omega}$, which is earth rotation rate. It will be further assumed that coordinate system "E" has its Y axis coincident with the spin axis of the earth, its Z axis in the equatorial plane passing through the Greenwich meridian, and its X axis in such a position as to form a right-handed set in the sequence XYZ.

The platform coordinate system will be denoted by the subscript "p", and its axes by x pointing east, y pointing north, and z outward along the local vertical. Figure 2 shows the E and p coordinate systems, together with vehicle latitude, Θ, and longitude, Λ. Since the vehicle is assumed to be located on the surface of the earth, the radius vector $\underline{R}$ is earth radius. Accelerometer sensitive axes are the x and y axes.

It will be assumed for the moment that a third accelerometer is present, oriented along the z axis. Then the three accelerometers sense the vector non-gravitational acceleration acting on the platform, denoted by the vector $\underline{A}$. Thus $\underline{A}$ is equal to the second time derivative of $\underline{R}$ with respect to inertial space minus the mass attraction gravity vector, $\underline{g}_m$, pointing downward. In equation form, this relationship is

$$\underline{A} = \left(\frac{d^2\underline{R}}{dt^2}\right)_I - \underline{g}_m \quad . \tag{1}$$

The basic mechanization problem is, then, to solve for the vector $\underline{R}$ from equation 1. The time derivative of $\underline{R}$ with respect to the set of

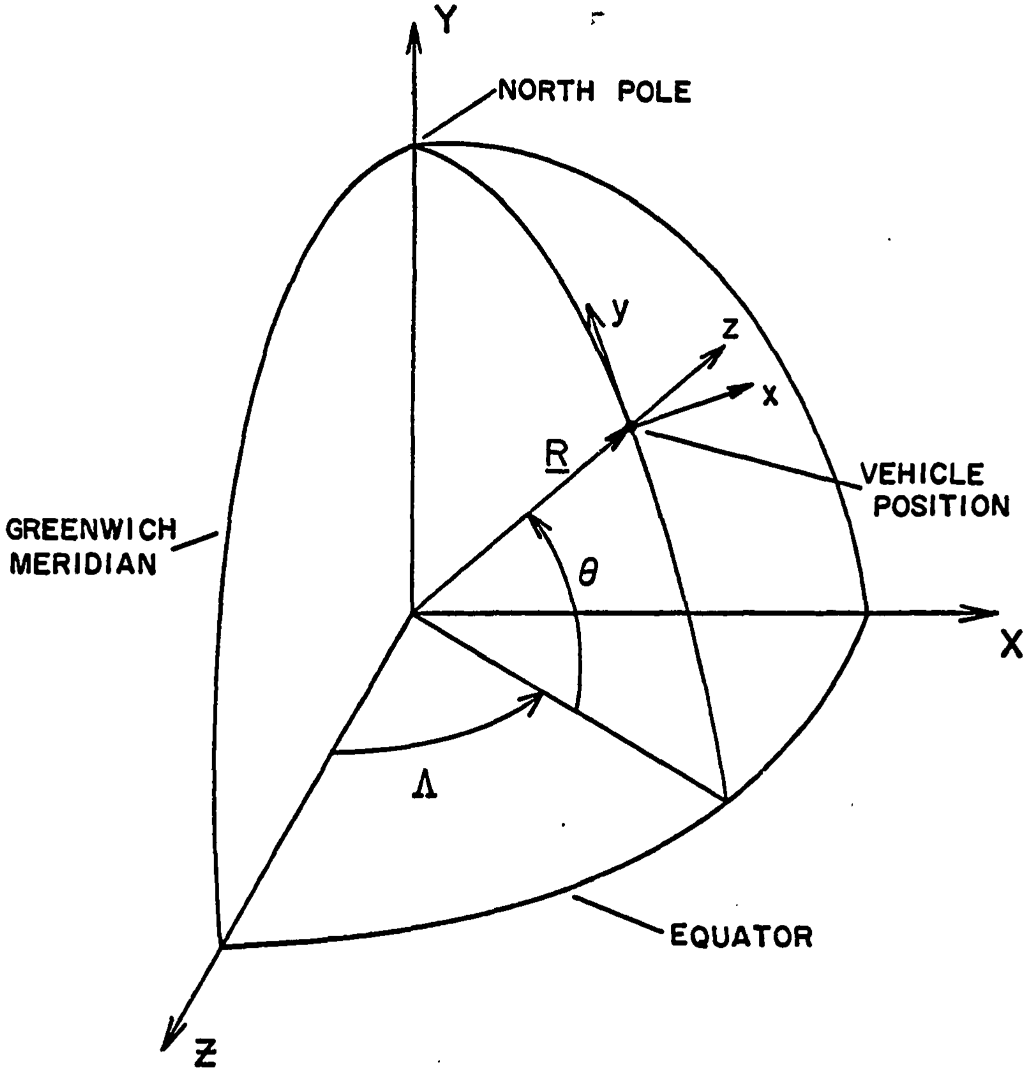

Figure 2. Orientation of earth-fixed and platform coordinate systems for latitude-longitude mechanization

earth-fixed axes E is vehicle velocity $\underline{V}$:

$$\left(\frac{d\underline{R}}{dt}\right)_E = \underline{V} \quad . \tag{2}$$

Then, by the well-known relation between time derivatives of a vector with respect to two coordinate systems with relative angular velocity, the time derivative of $\underline{R}$ relative to I can be written using equation 2 as

$$\begin{aligned}\left(\frac{d\underline{R}}{dt}\right)_I &= \left(\frac{d\underline{R}}{dt}\right)_E + \underline{\Omega} \times \underline{R} \\ &= \underline{V} + \underline{\Omega} \times \underline{R} \quad .\end{aligned} \tag{3}$$

Substitution of equation 3 into equation 1 yields

$$\underline{A} = \left(\frac{d\underline{V}}{dt}\right)_I + \left(\frac{d}{dt}\,(\underline{\Omega} \times \underline{R})\right)_I - \underline{g}_m \quad . \tag{4}$$

The angular rate (with respect to inertial space) of the platform, or p set of axes is defined as $\underline{\omega}$. Utilization of this, the constancy of $\underline{\Omega}$ and equation 3 results in equation 4 taking the form

$$\begin{aligned}\underline{A} &= \left(\frac{d\underline{V}}{dt}\right)_p + \underline{\omega} \times \underline{V} + \underline{\Omega} \times \left(\frac{d\underline{R}}{dt}\right)_I - \underline{g}_m \\ &= \left(\frac{d\underline{V}}{dt}\right)_p + \underline{\omega} \times \underline{V} + \underline{\Omega} \times \underline{V} + \underline{\Omega} \times (\underline{\Omega} \times \underline{R}) - \underline{g}_m \quad ,\end{aligned}$$

or

$$\underline{A} = \left(\frac{d\underline{V}}{dt}\right)_p + (2\underline{\Omega} + \underline{\rho}) \times \underline{V} - \underline{g} \quad , \tag{5}$$

where $\underline{\rho}$ is the angular velocity of p relative to E, $\underline{g} = \underline{g}_m - \underline{\Omega} \times (\underline{\Omega} \times \underline{R})$ is the "plumb-bob" gravity vector and $\left(\frac{d\underline{V}}{dt}\right)_p$ is the time derivative of $\underline{V}$ in the p coordinate system. From equation 2, the time derivative of $\underline{R}$ with respect to p can be written

$$\left(\frac{d\underline{R}}{dt}\right)_E = \left(\frac{d\underline{R}}{dt}\right)_p + \underline{\rho} \times \underline{R} \text{ , or}$$

$$\underline{V} = \left(\frac{d\underline{R}}{dt}\right)_p + \underline{\rho} \times \underline{R} \quad . \tag{6}$$

In terms of components along platform axes, denoted by the unit vectors $\underline{1}_x$, $\underline{1}_y$ and $\underline{1}_z$, $\left(\frac{d\underline{V}}{dt}\right)_p$ and $\left(\frac{d\underline{R}}{dt}\right)_p$ are given, respectively, as

$$\left(\frac{d\underline{V}}{dt}\right)_p = \dot{V}_x \underline{1}_x + \dot{V}_y \underline{1}_y + \dot{V}_z \underline{1}_z \quad , \tag{7}$$

and

$$\left(\frac{d\underline{R}}{dt}\right)_p = \dot{R}_x \underline{1}_x + \dot{R}_y \underline{1}_y + \dot{R}_z \underline{1}_z \quad . \tag{8}$$

Equation 5 may now be written in component form as

$$\begin{aligned} A_x &= \dot{V}_x + (2\Omega_y + \rho_y)V_z - (2\Omega_z + \rho_z)V_y - g_x \\ A_y &= \dot{V}_y + (2\Omega_z + \rho_z)V_x - (2\Omega_x + \rho_x)V_z - g_y \\ A_z &= \dot{V}_z + (2\Omega_x + \rho_x)V_y - (2\Omega_y + \rho_y)V_x - g_z \quad , \end{aligned} \tag{9}$$

and equation 6 as

$$\begin{aligned} V_x &= \dot{R}_x + \rho_y R_z - \rho_z R_y \\ V_y &= \dot{R}_y + \rho_z R_x - \rho_x R_z \\ V_z &= \dot{R}_z + \rho_x R_y - \rho_y R_x \quad . \end{aligned} \tag{10}$$

Up to this point, the equations presented are general in form and apply to any coordinate system. They will now be specialized to the latitude-longitude coordinate system described earlier and shown in

Figure 2. From Figure 2 it is apparent that $R_x = R_y = 0$ and $R_z = R =$ earth radius, a constant to within ellipticity and tidal variations in this application. Also, since $\underline{\Omega}$ is coincident with the Y axis and x always points east, $\Omega_x = 0$, $\Omega_y = \Omega \cos \Theta$ and $\Omega_x = \Omega \sin \Theta$. To within ellipticity and centripetal acceleration, $g_x = g_y = 0$ and $g_z = -g$. Such effects must be taken into account in the mechanization of the system under consideration, but they are neglected here because of the complexity of the terms and because they are not important in the basic problem being studied here. The interested reader is referred to Streeter (3), who presents mechanization equations for the latitude-longitude coordinate system and several others, including oblateness and centripetal terms.

Since a marine system is being studied here, it is also assumed that $V_z = \dot{V}_z = 0$. Further examination of Figure 2 reveals that $\dot{\Lambda} = \rho_y \cos \Theta + \rho_z \sin \Theta$ and that the component of $\underline{\rho}$ normal to x and Y is zero. Thus $\rho_y \sin \Theta = \rho_z \cos \Theta$, or

$$\rho_z = \rho_y \tan \Theta \quad . \tag{11}$$

Longitude rate, $\dot{\Lambda}$, may then be written

$$\begin{aligned} \dot{\Lambda} &= \rho_y (\cos \Theta + \frac{\sin^2 \Theta}{\cos \Theta}) \\ &= \rho_y (\frac{\cos^2 \Theta + \sin^2 \Theta}{\cos \Theta}) \\ &= \rho_y \sec \Theta \quad . \end{aligned} \tag{12}$$

Also from Figure 2,

$$\dot{\Theta} = -\rho_x \quad , \tag{13}$$

$$\Omega_y = \Omega \cos \Theta \quad , \tag{14}$$

and $\Omega_z = \Omega \sin \Theta$, (15)

as was stated previously. Equations 9 now become

$$A_x = \dot{V}_x - (2\Omega_x + \rho_z)V_y$$
$$A_y = \dot{V}_y + (2\Omega_z + \rho_z)V_x \quad . \qquad (16)$$

(The third of equations 9 may be ignored.) Equations 10 become

$$\rho_y = \frac{V_x}{R}$$
$$\rho_x = \frac{-V_y}{R} \quad . \qquad (17)$$

(The third of these equations may also be ignored.) Since the angular velocity of p relative to inertial space is $\underline{\omega} = \underline{\Omega} + \underline{\rho}$, gyro torquing rates are computed from

$$\omega_x = \rho_x$$
$$\omega_y = \Omega_y + \rho_y \qquad (18)$$
$$\omega_z = \Omega_z + \rho_z \quad .$$

Equations 11 through 18 are a complete set of mechanization equations for the latitude-longitude coordinate system which must be solved by the computer. Inputs are A_x and A_y from the accelerometers (the z-accelerometer is clearly not necessary); initial conditions Θ_o, Λ_o, V_{xo} and V_{yo}; and constants R and Ω. This set represents 12 equations in 12 unknowns: Λ, Θ, V_x, V_y, Ω_y, Ω_z, ω_x, ω_y, ω_z, ρ_x, ρ_y and ρ_z; thus solution is possible. In addition, of course, the initial platform orientation must be established.

If a celestial tracking device is employed, it is necessary to compute the orientations of the lines-of-sight to the celestial bodies being tracked, relative to the computed position of the platform. This is done quite simply from a knowledge of the Greenwich Hour Angles (GHA) and declinations (DEC) of the bodies. These quantities are shown in Figure 3, which is a diagram of the so-called celestial sphere, a sphere concentric with the earth. The GHA is measured westward from the plane of the Greenwich meridian, about the Y axis. The DEC, or "latitude" of the body, is measured as shown about an axis orthogonal to the plane containing the Y axis and the line-of-sight. The direction cosines between the line-of-sight (LOS) and the XYZ axes are

$$
\begin{aligned}
C_{LOS-X} &= -\cos(DEC)\sin(GHA) \\
C_{LOS-Y} &= \sin(DEC) \\
C_{LOS-Z} &= \cos(DEC)\cos(GHA)\ .
\end{aligned}
\tag{19}
$$

From Figure 2, the direction cosines relating the p and E coordinate systems may be written:

$$
\begin{aligned}
C_{Xx} &= \cos\Lambda \\
C_{Xy} &= -\sin\Theta\sin\Lambda \\
C_{Xz} &= \cos\Theta\sin\Lambda \\
C_{Yx} &= 0 \\
C_{Yy} &= \cos\Theta \\
C_{Yz} &= \sin\Theta \\
C_{Zx} &= -\sin\Lambda \\
C_{Zy} &= -\sin\Theta\cos\Lambda \\
C_{Zz} &= \cos\Theta\cos\Lambda\ .
\end{aligned}
\tag{20}
$$

The direction cosines of the LOS relative to p may now be written from the general relation

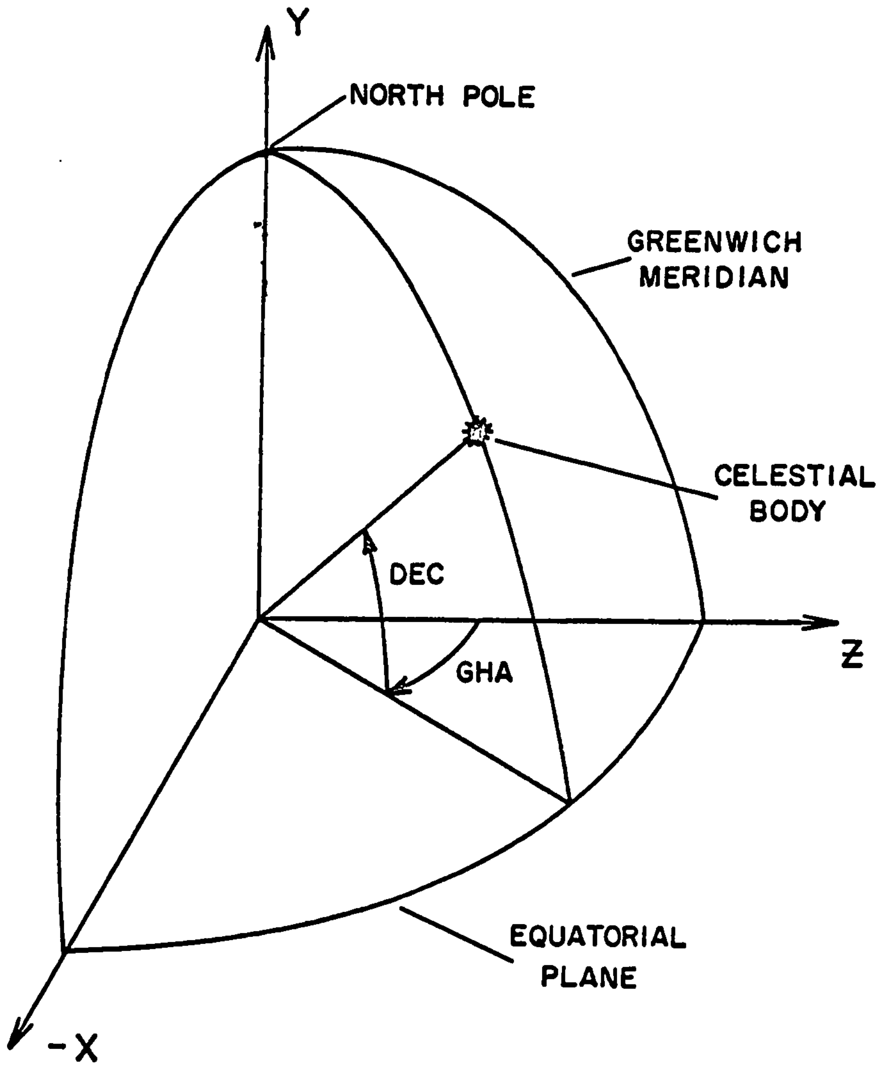

Figure 3. Celestial sphere and earth-fixed coordinate system

$$C_{LOS\ i} = C_{LOS-X}\, C_{Xi} + C_{LOS-Y}\, C_{Yi} + C_{LOS-Z}\, C_{Zi}\ , \tag{21}$$

where i = x, y, z. Equation 21 yields

$$\begin{aligned} C_{LOS\ x} &= -\cos(\text{DEC}) \sin(\text{GHA}) \cos\Lambda \\ &\quad - \cos(\text{DEC}) \cos(\text{GHA}) \sin\Lambda \\ C_{LOS\ y} &= \cos(\text{DEC}) \sin(\text{GHA}) \sin\Theta \sin\Lambda \\ &\quad + \sin(\text{DEC}) \cos\Theta \\ &\quad - \cos(\text{DEC}) \cos(\text{GHA}) \sin\Theta \cos\Lambda \\ C_{LOS\ z} &= -\cos(\text{DEC}) \sin(\text{GHA}) \cos\Theta \sin\Lambda \\ &\quad + \sin(\text{DEC}) \sin\Theta \\ &\quad + \cos(\text{DEC}) \cos(\text{GHA}) \cos\Theta \cos\Lambda\ . \end{aligned} \tag{22}$$

From these quantities the azimuth and altitude of the LOS may be readily computed. For example, if azimuth angle, A, is measured in a positive sense about z, it may be found from the relation

$$\tan A = \frac{(-\ C_{LOS\ x})}{C_{LOS\ y}}\ , \tag{23}$$

which avoids ambiguities as to quadrant. Altitude, η, or angle from the xy plane, can be determined from the simple relation

$$\sin\eta = C_{LOS\ z}\ . \tag{24}$$

In the next Section will be presented error propagation equations based on the mechanization equations presented here. Velocity damping techniques will also be presented there, since they are tied in so closely with error propagation.

III. ERROR PROPAGATION EQUATIONS

Errors in an inertial navigation system can result from many different sources. Some of these are initial position and velocity errors, initial alignment errors, computational errors, accelerometer bias and scale factor errors (bias being an error in the null point), extraneous torques which cause gyro drift rates, gyro torquing scale factor errors, and gyro and accelerometer misalignment errors with respect to the platform. In this section will be presented the equations describing the propagation of some of these errors into system position, velocity and alignment errors.

Throughout this section, it will be assumed that all errors are so small that only first-order effects need be considered. It will also be assumed that misalignment errors are so small that such angles may be represented as vectors, the misalignment being a rotation about that vector of the magnitude of the vector. (This assumption is really the same as the first.) Since a marine system is being considered, vehicle accelerations and velocities will be sufficiently small that accelerometer scale factor errors may be ignored, and platform angular rate $\underline{\omega}$ may be taken as earth rate $\underline{\Omega}$, a constant. (It is also tacitly assumed here that the ship is not operating near the north pole, so ρ_z is not large. A latitude of 45° is a convenient and reasonable choice which, from equations 11 and 17, sets $\rho_z = \rho_y = \frac{V_x}{R}$). It will be assumed that the computer operates perfectly. Generally, one of the first steps in the design of an inertial navigation system is the design of a computer which operates with sufficient accuracy to give negligible system errors. Instrument misalignments relative to the platform will also be neglected, as these can generally be determined and compensated with sufficient accuracy.

The errors to be considered, then, are accelerometer bias errors, gyro drift rate and torque scale factor errors, and initial condition errors. First, the error equations for a pure inertial system will be presented. Then velocity damping methods will be discussed, and error equations for this mode will be presented. An additional error source, external velocity error, will be introduced here. Error propagation in a damped-stellar-inertial mode will also be discussed, and automatic alignment techniques will be touched on briefly.

It is convenient to introduce three right-handed, orthogonal coordinate systems at this point. These will be defined by their unit vectors:

$\underline{1}_{xp}$, $\underline{1}_{yp}$, $\underline{1}_{zp}$ - platform coordinates, along the instrument axes, denoted by p.

$\underline{1}_{x}$, $\underline{1}_{y}$, $\underline{1}_{z}$ - true coordinates, representing ideal system alignment with no errors.

$\underline{1}_{xc}$, $\underline{1}_{yc}$, $\underline{1}_{zc}$ - computer coordinates, representing the alignment of the true set if computer position errors were zero.

If all errors were zero, these three coordinate systems would be coincident.

The computer, or c, coordinate system may seem artificial but serves a very useful purpose in what follows. This system, in fact, is the coordinate system in which the computer solves the equations of motion (equations 9), since the c system is the only one known to the computer.

The following vectors are now defined for use in writing the error equations:

$\underline{\omega}$ - angular velocity of the true system with respect to I.

$\underline{\omega}_c$ - angular velocity of the c system with respect to I.

$\underline{\omega}_p$ - angular velocity of the p system with respect to I.

$\underline{\rho}$ - angular velocity of the true system with respect to E.

Thus $\underline{\omega} = \underline{\Omega} + \underline{\rho}$.

$\delta\underline{\Theta}$ - vector angle between c and the true system.

$\underline{\Psi}$ - vector angle between p and c.

$\underline{\Phi}$ - vector angle between p and the true system.

Thus $\underline{\Phi} = \underline{\Psi} + \delta\underline{\Theta}$

These vector angles are the (small) misalignment angles between systems, oriented such that, for example, a rotation of the true system of amount $|\delta\underline{\Theta}|$ about the direction of $\delta\underline{\Theta}$ would yield the c system. As mentioned earlier, these misalignment angles may be characterized as vectors because of their assumed small magnitudes (of the order of minutes or arc at the most).

The error propgation equation for the angle $\underline{\Psi}$ will be presented first, since it is used in determing other system error equations and is so important to the later parts of this study. To begin, it is observed that the computer torques the gyros in an effort to cause the platform to rotate at an angular velocity $\underline{\omega}$. The best estimate of $\underline{\omega}$ available to the computer is, however, $\underline{\omega}_c$. This is not the true rate of rotation of the platform because of two additional terms. The first is an additional rotation rate due to gyro drift rate, $\underline{\varepsilon}'$, and torque scale factor errors, k_x, k_y and k_z. This total effect may be characterized as a drift rate $\underline{\varepsilon}$, where

$$\underline{\varepsilon} = \underline{\varepsilon}' + K\underline{\omega} \quad , \tag{25}$$

K is a diagonal matrix with scale factor errors along the diagonal, and

$\underline{\omega}$ is used in order to retain only first-order effects ($K(\underline{\omega}_c - \underline{\omega})$ is second-order). The second term arises because c and p differ by the angle $\underline{\Psi}$. The components of $\underline{\omega}_c$ are thus applied to the p axes, giving a platform rate of $\underline{\omega}_c$ rotated through the angle $\underline{\Psi}$, or, to first order, $\underline{\omega}_c + \underline{\Psi} \times \underline{\omega}_c$. The platform rate, $\underline{\omega}_p$, may now be written

$$\underline{\omega}_p = \underline{\varepsilon} + \underline{\omega}_c + \underline{\Psi} \times \underline{\omega}_c \quad . \tag{26}$$

It is also observed that, by definition, the difference between $\underline{\omega}_p$ and $\underline{\omega}_c$ is the time rate of change of $\underline{\Psi}$ with respect to c. This may be written

$$\underline{\omega}_p = \underline{\omega}_c + \left(\frac{d}{dt}\underline{\Psi}\right)_c \quad . \tag{27}$$

Equations 26 and 27 may now be combined to yield

$$\underline{\omega}_c + \left(\frac{d\underline{\Psi}}{dt}\right)_c = \underline{\varepsilon} + \underline{\omega}_c + \underline{\Psi} \times \underline{\omega}_c \quad ,$$

or

$$\left(\frac{d\underline{\Psi}}{dt}\right)_c + \underline{\omega}_c \times \underline{\Psi} = \underline{\varepsilon} \quad . \tag{28}$$

Use of the relation between time derivatives of a vector in two coordinate systems with relative angular velocity allows equation 28 to be written, with respect to the inertially fixed coordinate system I as

$$\left(\frac{d\underline{\Psi}}{dt}\right)_I = \underline{\varepsilon} \quad . \tag{29}$$

Use of this same relation allows determination of the $\underline{\Psi}$ equation in the true coordinate system:

$$\dot{\underline{\Psi}} + \underline{\omega} \times \underline{\Psi} = \underline{\varepsilon} \quad , \tag{30}$$

where, by definition,

$$\underline{\dot{\Psi}} = \left(\frac{d\underline{\Psi}}{dt}\right)_{true} \quad . \tag{31}$$

It should be noted that the error angle $\underline{\Psi}$ depends only on gyro drift rate $\underline{\varepsilon}$, and is independent of other system errors, except for initial misalignments.

A physical interpretation of $\underline{\Psi}$ will now be developed. Figure 4 shows the earth's surface, the true and computed positions and the true and computer coordinate systems. For clarity, only a latitude error has been assumed, and it has been exaggerated. The computer coordinate system is parallel to the locally level system shown at the computer position. Consider now the computation of the angle of the line-of-sight to a star in the xy plane. The computer makes this computation (described in Section II) based on computed position and a correctly-aligned coordinate system located there. The angle computed is μ_c, shown in Figure 4. If a telescope is used to measure this angle, and the computer and platform coordinate systems coincide, the angle measured, μ_o in Figure 4, will be equal to μ_c, because stars are essentially at an infinite distance. (The nearest star is about 4.5 light-years, or 2.4×10^{13} miles distant, which corresponds to a parallax error of about 10^{-8} seconds of arc per mile, a completely negligible factor.) The situation above corresponds to $\underline{\Psi} = 0$, and it is thus apparent that the $\underline{\Psi}$ angle represents the telescope pointing angle error, here and in the general case. It is also apparent that use of a telescope allows determination of $\underline{\Psi}$ (in the case of observation of two stars) or components of $\underline{\Psi}$, but not $\delta\underline{\Theta}$. The sun is about 93×10^6 miles distant; the parallax error in this case is about 2×10^{-3}

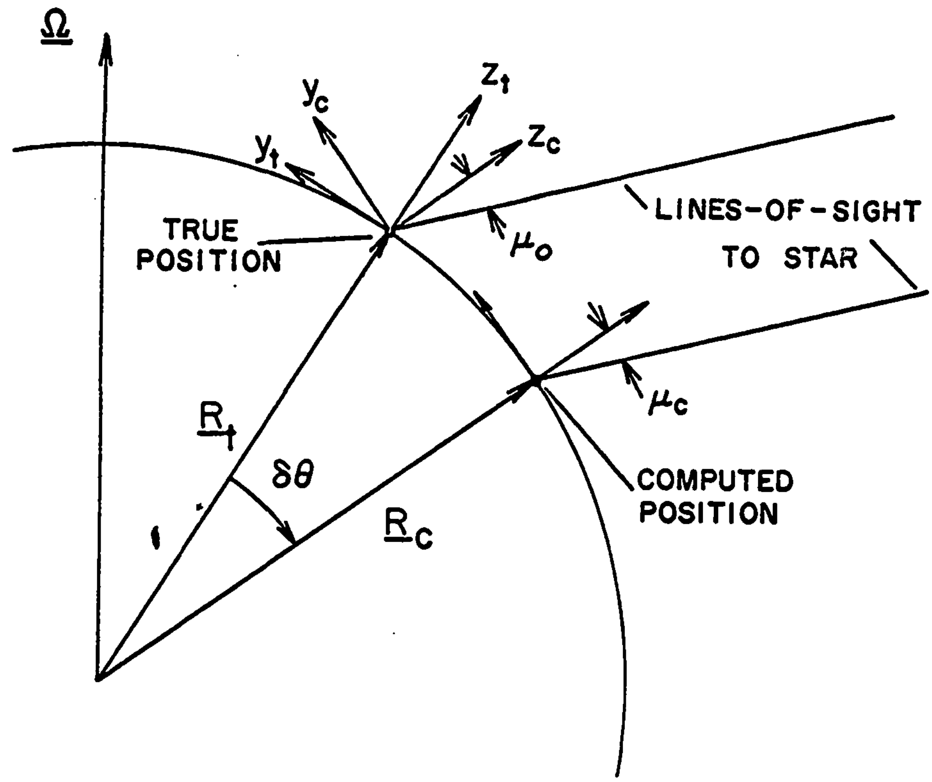

Figure 4. Computer and true coordinate systems, and line-of-sight to a star

seconds of arc per mile, which is also negligible. Thus sun-tracking, which is the subject of this study, will also yield only components of the $\underline{\Psi}$ angle, and no information as to $\delta\underline{\Theta}$.

System position error equations may be derived from equation 1, or a similar equation, which must be solved by the computer to determine system position. It is more convenient to write equation 1 in terms of time derivatives with respect to the c coordinate system:

$$\left(\frac{d^2\underline{R}}{dt^2}\right)_c + 2\,\underline{\omega}_c \times \left(\frac{d\underline{R}}{dt}\right)_c + \left(\frac{d}{dt}\,\underline{\omega}_c\right)_c \times \underline{R}$$

$$+\ \underline{\omega}_c \times (\underline{\omega}_c \times \underline{R}) - \underline{\Omega} \times (\underline{\Omega} \times \underline{R})$$

$$= \underline{A} + \underline{g}\ , \tag{32}$$

where $\underline{g}$ is the "plumb-bob" gravity vector defined earlier. The errors made in solving this equation will now be discussed. First, the actual acceleration input to the computer differs from true acceleration $\underline{A}$ because of two error sources. The first is accelerometer bias error, represented by $\underline{\nabla}$; the second arises because the accelerometers measure $\underline{A}$ in the platform coordinate system and the computer operates on these outputs in the computer system. This results in the additional term $-\ \underline{\Psi} \times \underline{A}$. The term $\underline{A}$ in equation 32 must therefore be replaced by $\underline{A} - \underline{\Psi} \times \underline{A} + \underline{\nabla}$.

The gravity term in equation 32 is the value of $\underline{g}$ determined by the computer from $\underline{R}$. To within ellipticity (which will be a second-order effect in error equations), $\underline{g}$ may be written

$$\underline{g} = -\,\frac{g}{R}\,\underline{R}$$

$$= -\omega_o^{\,2}\,\underline{R}\ , \tag{33}$$

where ω_o^2, by definition, is $\frac{g}{R}$. The constant ω_o^2 is called the Schuler angular frequency, and corresponds to a period of about 84 minutes. Error in the computed value of $\underline{g}$ results from error in $\underline{R}$, $\delta\underline{R}$. Thus $\underline{g}$ must be replaced by $\underline{g} - \omega_o^2\ \delta\underline{R}$.

Each $\underline{R}$ in equation 32 must now be replaced by $\underline{R} + \delta\underline{R}$, and the additional error terms discussed above must be inserted. The values of $\underline{\omega}_c$ and $\underline{\Omega}$ are known exactly in the c system, so no errors are present there. The resulting error equation is

$$\left(\frac{d^2\delta\underline{R}}{dt^2}\right)_c + 2\ \underline{\omega}_c \times \left(\frac{d\delta\underline{R}}{dt}\right)_c + \left(\frac{d\underline{\omega}_c}{dt}\right)_c \times \delta\underline{R}$$

$$+ \underline{\omega}_c \times (\underline{\omega}_c \times \delta\underline{R}) - \underline{\Omega} \times (\underline{\Omega} \times \delta\underline{R})$$

$$= \underline{\nabla} - \underline{\Psi} \times \underline{A} - \omega_o^2\ \delta\underline{R} \quad . \tag{34}$$

This equation can be further simplified by making the approximations that $\underline{\omega}_c = \underline{\Omega}$ and that the total vehicle acceleration is zero for the present application, as discussed earlier. Equation 34 then becomes

$$\left(\frac{d^2\delta\underline{R}}{dt^2}\right)_c + 2\ \underline{\Omega} \times \left(\frac{d\delta\underline{R}}{dt}\right)_c + \omega_o^2\ \delta\underline{R}$$

$$= \underline{\nabla} + \underline{\Psi} \times \underline{g}\ , \tag{35}$$

since $\underline{A}$ is the non-gravitational acceleration vector. Equation 35 can be rewritten in terms of derivatives with respect to I, and then with respect to the true set to yield

$$\delta\ddot{\underline{R}} + 2\ \underline{\Omega} \times \delta\dot{\underline{R}} + \omega_o^2\ \delta\underline{R} = \underline{\nabla} + \underline{\Psi} \times \underline{g}\ , \tag{36}$$

where dots denote time derivatives with respect to the true system.

The x and y components of equation 36 are

$$\ddot{\delta R}_x - 2\,\Omega_z\,\dot{\delta R}_y + \omega_o^{\,2}\,\delta R_x = \nabla_x - g\Psi_y$$
$$\ddot{\delta R}_y + 2\,\Omega_z\,\dot{\delta R}_x + \omega_o^{\,2}\,\delta R_y = \nabla_y + g\Psi_x\ . \qquad (37)$$

The z-channel is again ignored. (It is interesting to note in passing that it is inherently unstable.) It should be noted that equations 37 are linear, constant-coefficient, coupled differential equations and can therefore be solved fairly easily. First, it is enlightening to determine the normal modes of equations 37. This can be done by taking Laplace transforms of the equations with all initial conditions and driving functions set equal to zero, then finding the zeroes of the system determinant. This results in the equation

$$(s^2 + \omega_o^{\,2})^2 + 4\,\Omega_z^2\,s^2 = 0\quad , \qquad (38)$$

with second-degree roots

$$s^2 = -(\omega_o^{\,2} + 2\,\Omega_z^2) \pm 2\,\Omega_z\sqrt{\omega_o^{\,2} + \Omega_z^2}\ . \qquad (39)$$

Since the maximum value of Ω_z is about $\frac{\omega_o}{10}$, second-order terms in Ω_z can be neglected. The result is

$$s^2 \doteq -\,\omega_o^{\,2} \pm 2\,\Omega_z\,\omega_o\quad , \qquad (40)$$

and it is clear that these roots are always negative. Thus the zeroes of s are purely imaginary and it follows that system errors propagate in an undamped or oscillatory fashion with approximately an 84-minute period.

It can be shown (Pinson, (2)) that the cross-coupling terms in equations 37 result merely in a slow precession of system errors about the z axis, and can therefore be ignored for short time periods. In fact, for a vehicle on the equator, $\Omega_z \equiv 0$ and no cross-coupling exists.

With this approximation, equations 37 become

$$\begin{aligned}\delta\ddot{R}_x + \omega_o^2\,\delta R_x &= \nabla_x - g\Psi_y \\ \delta\ddot{R}_y + \omega_o^2\,\delta R_y &= \nabla_y + g\Psi_x\ ,\end{aligned} \tag{41}$$

and it is quite apparent that undamped error propagation results.

Equation 30 also takes a simpler form when $\underline{\omega}$ is set equal to $\underline{\Omega}$:

$$\dot{\underline{\Psi}} + \underline{\Omega} \times \underline{\Psi} = \underline{\varepsilon}\ , \text{ or} \tag{42}$$

$$\begin{aligned}\dot{\Psi}_x + \Omega_y\Psi_z - \Omega_z\Psi_y &= \varepsilon_x \\ \dot{\Psi}_y + \Omega_z\Psi_x \qquad &= \varepsilon_y \\ \dot{\Psi}_z - \Omega_y\Psi_x \qquad &= \varepsilon_z\ .\end{aligned} \tag{43}$$

These also are linear, constant-coefficient differential equations. Their normal modes are found from the equation

$$s^3 + (\Omega_y^{\ 2} + \Omega_z^{\ 2})s = 0\ ,$$

or

$$s(s^2 + \Omega^2) = 0\ , \tag{44}$$

the roots being $s = 0, \pm j\Omega$.

Error propagation in the $\underline{\Psi}$ equation may be visualized more clearly by finding the components of equation 42 in the epq coordinate system, a system fixed with respect to the platform and with the e axis coincident with x, the p axis parallel to $\underline{\Omega}$ and the q axis completing the right-handed set, as shown in Figure 5. In this system, only the p axis has a component of $\underline{\Omega}$, and equation 42 may be written

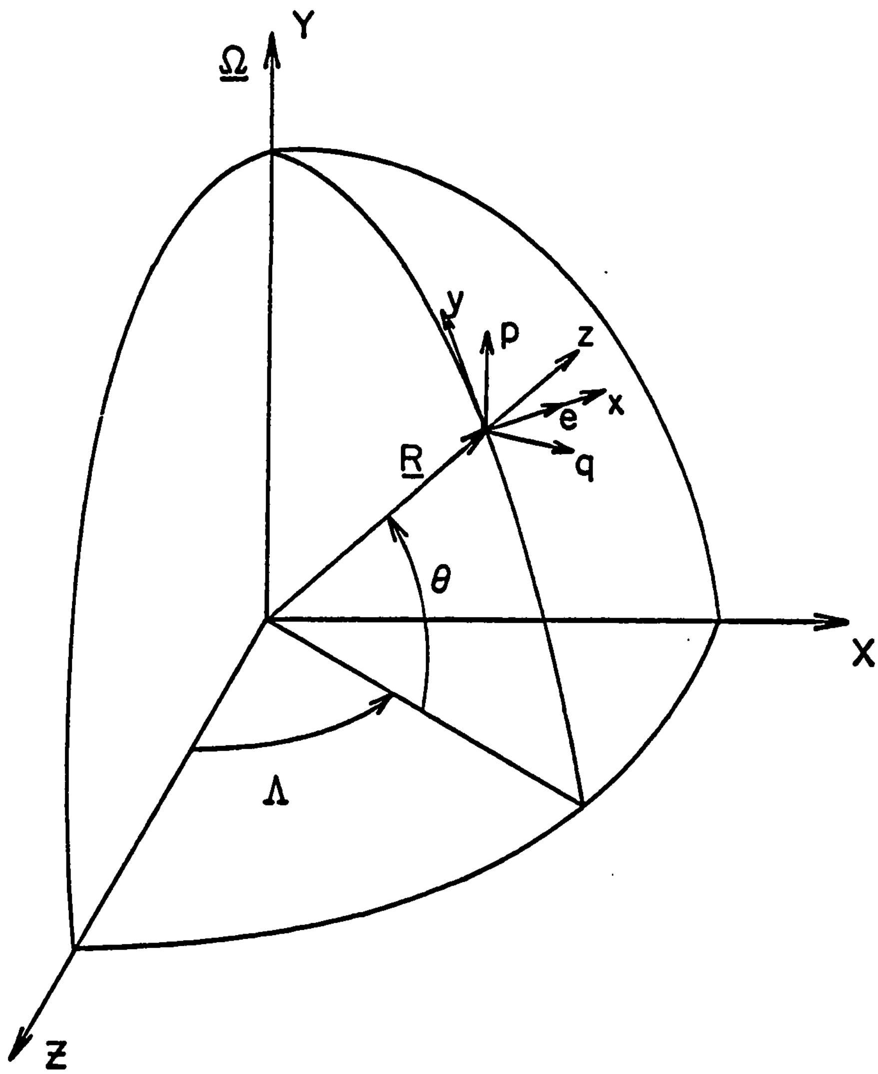

Figure 5. Platform (xyz) and epq coordinate systems

$$\dot{\Psi}_e + \Omega\Psi_q = \varepsilon_e$$
$$\dot{\Psi}_p = \varepsilon_p \qquad (45$$
$$\dot{\Psi}_q - \Omega\Psi_e = \varepsilon_q \quad .$$

The error propagation modes are now quite clear; the p channel is a simp integration while the e and q channels are coupled and undamped with a 2 hour oscillatory period.

In summary then, a pure inertial navigation system has error propag tion modes characterized by a simple integration and by undamped oscilla tory coupled modes with 84-minute and 24-hour periods. The 84-minute mo is a direct consequence of the inability of accelerometers to sense grav tational acceleration and may be visualized more clearly by considering simple, single-channel system located on the surface of the earth, as shown in Figure 6. It is assumed that the system is at rest, and that i is mechanized in a locally-level mode. It will now be assumed that the x-accelerometer has a small bias error ∇_x. The computer interprets this as an x-acceleration; thus a position error δR_x is generated. In order keep the platform "level", the computer supplies an erroneous torquing r $\frac{\delta \dot{R}_x}{R}$ to the y gyro. The platform angular tilt with respect to true local level is therefore $\frac{\delta R_x}{R}$. Now a component of gravitational acceleration exists along the x axis. The magnitude of this acceleration, for small tilt angles is $g \frac{\delta R_x}{R}$, in the positive x direction. The net acceleratio of the platform is, of course, zero; the non-gravitational acceleration the x direction is therefore $- \frac{g}{R} \delta R_x$. The total output of the x acceler meter is therefore $\nabla_x - \frac{g}{R} \delta R_x$, which in the simple system considered her

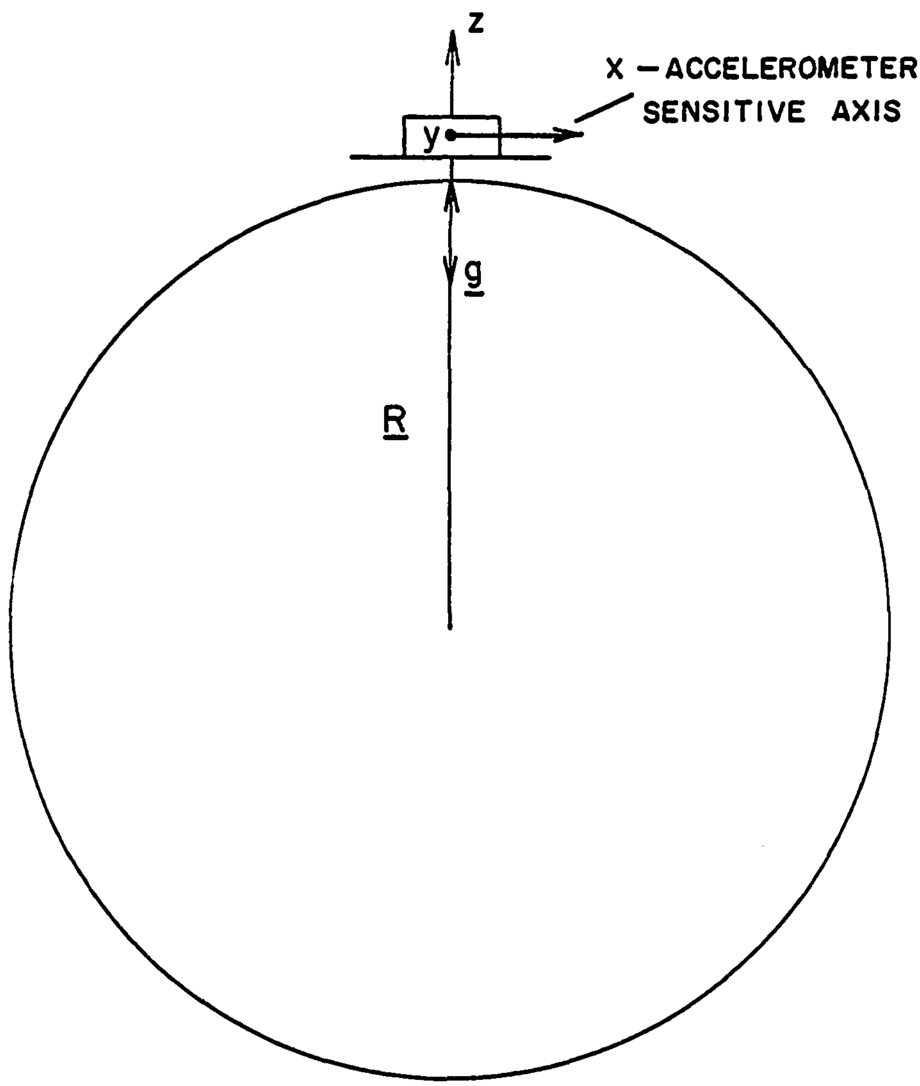

Figure 6. Single-channel inertial navigation system

is equal to $\delta\ddot{R}_x$. The error equation thus becomes

$$\delta\ddot{R}_x + \frac{g}{R}\,\delta R_x = \nabla_x \text{ , or}$$

$$\delta\ddot{R}_x + \omega_o^{\,2}\,\delta R_x = \nabla_x \text{ , and the}$$

84-minute or Schuler period oscillation is illustrated quite clearly. Although it is not as obvious, this 84-minute error mode can be shown to be present in all system mechanization schemes, as can be seen from equation 34, which is independent of the coordinate system used. This characteristic has the advantage of bounding system errors caused by such sources as constant accelerometer bias errors and initial velocity errors. Disadvantages are the facts that initial errors will never die out with time, and most random error sources will give rise to system errors with variances growing ultimately linearly with time, as in the classical random-walk process. (This characteristic will be discussed in more detail later.). It is because of these disadvantages that damped inertial system mechanizations, utilizing an external source of reference velocity were developed. One such mechanization and the resulting error equations will now be presented.

First the concept of dynamic exactness will be discussed. The mechanizations developed in Section II may be said to be dynamically exact in that, in the absence of errors, system position and velocity can be determined correctly, even in a dynamic environment. The basic problem here is to alter the mechanization equations so as to achieve damping of the 84-minute error modes. It is extremely desirable to achieve this without destroying the dynamic exactness of the mechanization; otherwise, system errors would in general be present even in the absence of initial or instrument errors. This may be accomplished by introduction into the mechanization equations of the difference between system velocity and velocity indicated by an external source, here, a ship's water speed indicator or log, for example. Dynamic exactness will be retained because, in the absence of errors, this quantity will be zero.

One of the simplest and most common methods of accomplishing this is the introduction of an additive term proportional to velocity difference, $K_1 \underline{V}_d$, into the left-hand side of equation 32, where K_1 is a constant and

$$\underline{V}_d = \underline{V} - \underline{V}_r \tag{46}$$

is the vector velocity difference. The vector $\underline{V}_r$ is the external or reference velocity. In this application, the error in $\underline{V}$, $\delta\underline{V}$, can be written simply

$$\delta\underline{V} = \delta\underline{\dot{R}} \quad . \tag{47}$$

The reference velocity error can be written simply as $\delta\underline{V}_r$. (There is also, strictly speaking, an error term $- \underline{\Psi} \times \underline{V}$, but $\underline{V}$ is assumed so small that this term can be neglected.) Equation 46 then becomes

$$\underline{V}_d = \delta\underline{\dot{R}} - \delta\underline{V}_r \ , \tag{48}$$

and is introduced directly into the error equation 36, which becomes

$$\delta\underline{\ddot{R}} + 2\underline{\Omega} \times \delta\underline{\dot{R}} + K_1\delta\underline{\dot{R}} + \omega_o^{\ 2}\delta\underline{R} = \underline{\nabla} + \underline{\Psi} \times \underline{g} + K_1\delta\underline{V}_r \ . \tag{49}$$

Equation 49, in component form is

$$\begin{aligned} \delta\ddot{R}_x - 2\Omega_z\delta R_y + K_1\delta\dot{R}_x + \omega_o^{\ 2}\delta R_x &= \nabla_x - g\Psi_y + K_1\delta V_{rx} \\ \delta\ddot{R}_y + 2\Omega_z\delta R_x + K_1\delta\dot{R}_y + \omega_o^{\ 2}\delta R_y &= \nabla_y + g\Psi_x + K_1\delta V_{ry}. \end{aligned} \tag{50}$$

Pinson (2) shows that, because of the term $K_1\delta\underline{\dot{R}}$ in equation 49 (with $K_1 > 0$), the normal modes of equations 50 are damped, and that the cross-coupling term $2\underline{\Omega} \times \delta\dot{\underline{R}}$ can therefore be ignored for all practical purposes without restrictions as to the time period considered. With this simplification, equations 50 become

$$\begin{aligned}
\delta\ddot{R}_x + K_1\delta\dot{R}_x + \omega_o^2\delta R_x &= \nabla_x - g\Psi_y + K_1\delta V_{rx} \\
\delta\ddot{R}_y + K_1\delta\dot{R}_y + \omega_o^2\delta R_y &= \nabla_y + g\Psi_x + K_1\delta V_{ry} \ ,
\end{aligned} \tag{51}$$

which are linear, constant-coefficient, uncoupled differential equations. Use of Laplace transforms results in solutions for δR_x and δR_y in the complex frequency or s-domain of

$$\begin{aligned}
\delta R_x(s) &= \frac{\nabla_x(s) - g\Psi_y(s) + 2\zeta\omega_o\delta V_{rx}(s) + \delta\dot{R}_{xo} + (s+2\zeta\omega_o)\delta R_{xo}}{s^2+2\zeta\omega_o s+\omega_o^2} \\
\delta R_y(s) &= \frac{\nabla_y(s) + g\Psi_x(s) + 2\zeta\omega_o\delta V_{ry}(s) + \delta\dot{R}_{yo} + (s+2\zeta\omega_o)\delta R_{yo}}{s^2+2\zeta\omega_o s+\omega_o^2}
\end{aligned} \tag{52}$$

where, by definition, $K_1 = 2\zeta\omega_o$, and δR_{xo}, δR_{yo}, $\delta\dot{R}_{xo}$ and $\delta\dot{R}_{yo}$ are initial condition errors. It is now quite obvious that system normal modes are damped, with damping ratio ζ and undamped natural frequency ω_o. Conventionally, ζ takes on values between 0.1 and 0.8, resulting in damped oscillatory modes. (It is noted in passing that an optimum value of ζ could be determined if error magnitudes for a particular system were known.)

Use of the final value theorem of Laplace transforms shows that system errors due to initial position and velocity errors settle to zero in steady-state. It is also noted that system errors due to stationary random inputs (a random component of ∇_x, for example, or δV_{rx}) have bounded variances.

The transfer function relating $\Psi_x(s)$ to $\delta R_y(s)$ can be written from equations 52 as

$$\frac{\delta R_y(s)}{\Psi_x(s)} = \frac{g}{s^2+2\zeta\omega_o s+\omega_o^2}$$
$$= R\frac{\omega_o^2}{s^2 + 2\zeta\omega_o s+\omega_o^2} \quad . \tag{53}$$

Equations 44 show that the normal modes of the $\underline{\Psi}$ equation are undamped oscillatory modes with a 24-hour period. Since ω_o corresponds to an 84-minuted period, Ψ_x (and Ψ_y) are seen to consist primarily of oscillatory time functions with a dominant frequency about $\frac{1}{16}$th that of the undamped natural frequency of the transfer function, equation 53. Thus, after initial transients have died out, the relation between $\delta R_y(s)$ and $\Psi_x(s)$ may be written, to a very good approximation, as merely

$$\frac{\delta R_y(s)}{\Psi_x(s)} = R \ . \tag{54}$$

Exactly the same approximation may be made for the x-channel. The result, then, for steady-state system errors is

$$\delta R_x(s) = -R\Psi_y(s) + \frac{\nabla_x(s)+2\zeta\omega_o\delta V_{rx}(s)}{s^2+2\zeta\omega_o s+\omega_o^2}$$
$$\delta R_y(s) = R\Psi_x(s) + \frac{\nabla_y(s)+2\zeta\omega_o\delta V_{ry}(s)}{s^2+2\zeta\omega_o s+\omega_o^2} \ . \tag{55}$$

Steady-state system position errors thus result from three sources for this application: accelerometer bias errors, reference velocity errors, and star pointing-angle errors, or $\underline{\Psi}$ angles. It will now be assumed that reference velocity and bias errors are constant. Equations 54 may then be written in the time domain as

$$\delta R_x(t) = -R\Psi_y(t) + \frac{R\nabla_x}{g} + \frac{2\zeta\delta V_{rx}}{\omega_o}$$
$$\delta R_y(t) = R\Psi_x(t) + \frac{R\nabla_y}{g} + \frac{2\zeta\delta V_{ry}}{\omega_o} \quad . \tag{56}$$

Thus, the only varying system errors are those due to $\underline{\Psi}$. These errors are of primary interest here, because they are ordinarily the dominant errors in such a system, and are the errors which can be corrected through use of a star- or sun-tracking device.

To find these errors, it is necessary to solve equation 42 for $\underline{\Psi}$. This can be done most conveniently by using equations 45, then finding Ψ_x and Ψ_y by means of the simple coordinate transformation:

$$\Psi_x = \Psi_e$$
$$\Psi_y = \Psi_p \cos\Theta - \Psi_q \sin\Theta \tag{57}$$
$$\Psi_z = \Psi_p \sin\Theta + \Psi_q \cos\Theta \quad .$$

Equations 45, in Laplace transform notation, are

$$\begin{pmatrix} s & \Omega \\ -\Omega & s \end{pmatrix}\begin{pmatrix} \Psi_e(s) \\ \Psi_q(s) \end{pmatrix} = \begin{pmatrix} \Psi_{eo} + \varepsilon_e(s) \\ \Psi_{qo} + \varepsilon_q(s) \end{pmatrix}$$
$$s\Psi_p(s) = \Psi_{po} + \varepsilon_p(s) \quad , \tag{58}$$

with solutions

$$\Psi_e(s) = \frac{1}{s^2 + \Omega^2}\left(s(\Psi_{eo} + \varepsilon_e(s)) - \Omega(\Psi_{qo} + \varepsilon_q(s))\right)$$

$$\Psi_q(s) = \frac{1}{s^2 + \Omega^2}\left(s(\Psi_{qo} + \varepsilon_q(s)) + \Omega(\Psi_{eo} + \varepsilon_e(s))\right) \tag{59}$$

$$\Psi_p(s) = \frac{1}{s}\left(\Psi_{po} + \varepsilon_p(s)\right) \quad ,$$

or, in the time domain (using the convolution theorem),

$$\begin{aligned}
\Psi_e(t) &= \Psi_{eo} \cos \Omega t - \Psi_{qo} \sin \Omega t \\
&\quad + \int_0^t [\varepsilon_e(u) \cos \Omega(t-u) - \varepsilon_q(u) \sin \Omega(t-u)[du \\
\Psi_q(t) &= \Psi_{qo} \cos \Omega t + \Psi_{eo} \sin \Omega t \\
&\quad + \int_0^t [\varepsilon_q(u) \cos \Omega(t-u) + \varepsilon_e(u) \sin \Omega(t-u)]du \\
\Psi_p(t) &= \Psi_{po} + \int_0^t \varepsilon_p(u)du \quad ,
\end{aligned} \tag{60}$$

where Ψ_{eo}, Ψ_{po} and Ψ_{qo} are initial values of the components of $\underline{\Psi}$.

Rather than resolving these angles immediately through equations 57, it is more convenient to develop expressions for system latitude, longitude, tilt and azimuth errors, then solve for these in terms of $\underline{\Psi}$. The components of the error angle $\delta\underline{\Theta}$, between computer and true axes, in the level plane are by definition

$$\begin{aligned}
\delta\Theta_x &= -\frac{\delta R_y}{R} \\
\delta\Theta_y &= \frac{\delta R_x}{R} \quad .
\end{aligned} \tag{61}$$

Components of $\underline{\Phi}$ in the level plane are, by definition also,

$$\begin{aligned}
\Phi_x &= \Psi_x + \delta\Theta_x = \Psi_x - \frac{\delta R_y}{R} \\
\Phi_y &= \Psi_y + \delta\Theta_y = \Psi_y + \frac{\delta R_x}{R} \quad ,
\end{aligned} \tag{62}$$

where equation 61 was also used. Since $\underline{\Phi}$ is the angle between the true axes and the platform, it follows that Φ_x and Φ_y represent platform tilts with respect to the local vertical, and that Φ_z represents the north-pointing, or azimuth error of the platform. By definition,

$$\Phi_z = \Psi_z + \delta\Theta_z \ . \tag{63}$$

In order to find azimuth error, then, $\delta\Theta_z$ must be found. Since the computer coordinate system is perfectly aligned at the computed position, it follows that the computer set is obtained by rotating the true set through angle $-\Delta\Theta$ about the x axis and $\Delta\Lambda$ about the p axis ($\Delta\Theta$ and $\Delta\Lambda$ being latitude and longitude errors, respectively). The components of $\delta\underline{\Theta}$ can therefore be written

$$\begin{aligned} \delta\Theta_x &= -\Delta\Theta \\ \delta\Theta_y &= \Delta\Lambda \cos\Theta \\ \delta\Theta_z &= \Delta\Lambda \sin\Theta \ . \end{aligned} \tag{64}$$

Then, from the first two of equations 64, and equations 61,

$$\begin{aligned} \Delta\Theta &= \frac{\delta R_y}{R} \\ \Delta\Lambda &= \frac{\delta R_x}{R\cos\Theta} \ . \end{aligned} \tag{65}$$

The third of equations 64 can now be written, using equations 65 as

$$\delta\Theta_z = \frac{\delta R_x}{R}\tan\Theta \ , \tag{66}$$

and azimuth error, Φ_z, from equations 63 and 66 becomes

$$\Phi_z = \Psi_z + \frac{\delta R_x}{R}\tan\Theta \ . \tag{67}$$

Use of equations 56, 57, 62, 65 and 67 gives the following set:

$$\Delta\Theta(t) = \Psi_e(t) + \frac{\Delta_y}{g} + \frac{2\zeta\delta V_{ry}}{\omega_o R} \tag{68}$$

$$\Delta\Lambda(t) = -\Psi_p(t) + \Psi_q(t) \tan \Theta$$

$$+ \frac{1}{\cos \Theta} \left(\frac{\Delta_x}{g} + \frac{2\zeta\delta V_{rx}}{\omega_o R} \right) \tag{69}$$

$$\Phi_z(t) = \frac{\Psi_q(t)}{\cos \Theta} + \tan \Theta \left(\frac{\Delta_x}{g} + \frac{2\zeta\delta V_{rx}}{\omega_o R} \right) \tag{70}$$

$$\Phi_x(t) = -\frac{\Delta_y}{g} - \frac{2\zeta\delta V_{ry}}{\omega_o R} \tag{71}$$

$$\Phi_y(t) = \frac{\Delta_x}{g} + \frac{2\zeta\delta V_{rx}}{\omega_o R} \quad , \tag{72}$$

where $\Psi_e(t)$, $\Psi_p(t)$ and $\Psi_q(t)$ are given by equations 59. Equations 68 and 69 are of general interest in any application, as they give system position errors. Equations 70, 71 and 72 may be of extreme interest in certain applications, such as the use of a ship as a missile-launching platform. Equation 70 is also of general interest in most cases, in that it gives the system azimuth or steering error.

If gyro drift rates are assumed constant in equations 59, the solutions of these equations obviously yield bounded sinusoidal variations in Ψ_e and Ψ_q and a linear increase in Ψ_p:

$$\Psi_e(t) = \Psi_{eo} \cos \Omega t - \Psi_{qo} \sin \Omega t$$

$$+ \frac{\varepsilon_e}{\Omega} \sin \Omega t - \frac{\varepsilon_q}{\Omega} (1 - \cos \Omega t) \tag{73}$$

$$\Psi_q(t) = \Psi_{qo} \cos \Omega t + \Psi_{eo} \sin \Omega t$$

$$+ \frac{\varepsilon_q}{\Omega} \sin \Omega t + \frac{\varepsilon_e}{\Omega} (1 - \cos \Omega t) \tag{74}$$

$$\Psi_p(t) = \Psi_{po} + \varepsilon_p t \quad . \tag{75}$$

Examination of equations 68, 69 and 70 then shows that latitude and azimuth errors will be bounded, and longitude error will increase without bound. On the other hand, if $\varepsilon_e(t)$, $\varepsilon_p(t)$ and $\varepsilon_q(t)$ are assumed to be stationary random functions, it can be shown (see Appendix A) that Ψ_e, Ψ_p and Ψ_q will all, in general, become random, non-stationary functions with variances increasing with time in an unbounded fashion. Thus, latitude, longitude and azimuth errors will also possess variances which grow in a similar manner.

This is the most serious drawback of a damped inertial navigation system, since gyros are experimentally found to have components of drift rate which are random. These rates, through the undamped 24-hour mode, result in system errors which can become intolerably large, statistically, for long operating times. It is thus necessary to find some means of damping the $\underline{\Psi}$ equation.

The simplest such method is that of obtaining position information periodically from some external source. Equations 68 and 69 indicate that, to within bias and reference velocity errors, latitude and longitude errors provide a direct measure of components of $\underline{\Psi}$. Through periodic position checks, then, and the use of the $\underline{\Psi}$ equation, estimates of all of the components of $\underline{\Psi}$ can be determined and corrections applied. Such a scheme requires the presence of landmarks or other check-points, however, which are not common in mid-ocean areas. A star-tracker capable of tracking two or more stars can provide a direct measure of the instantaneous value of $\underline{\Psi}$, and corrections can be made quite directly. Such a system, in effect, replaces the $\underline{\Psi}$ error with star-tracker pointing errors, which are ordinarily stationary. Thus all system errors would be bounded. Clear weather is required in order to implement this scheme.

A third scheme makes use of the fact that system velocity errors contain information as to components of $\underline{\Psi}$. In fact, differentiation of equations 56 yields (assuming other error quantities are constants)

$$\begin{aligned} \delta\dot{R}_x &= -R\dot{\Psi}_y \\ \delta\dot{R}_y &= R\dot{\Psi}_x \; . \end{aligned} \tag{76}$$

It is then possible to use this information in the gyro torquing process to damp the $\underline{\Psi}$ equations. In order to retain dynamic exactness, the velocity difference signals are used, these being

$$\begin{aligned} V_{dx} &= \delta\dot{R}_x - \delta V_{rx} \\ V_{dy} &= \delta\dot{R}_y - \delta V_{ry} \; . \end{aligned} \tag{77}$$

Although such a scheme is commonly used to achieve automatic system alignment (automatic leveling and gyrocompassing), it is not generally used during navigation because reference velocity errors produce undesirably large system errors. Pitman (1) and Pinson (2) both have excellent discussions of such automatic alignment mechanizations. A variation on such a mechanization is found to be required for the solution of the problem being studied here, and is derived in the next section.

A sun-tracking device, which is being considered in this study, would perform a function very similar to that of a star-tracker. As was pointed out earlier, a sun-tracker can be designed to operate under virtually any weather conditions. However, it can only detect the component of $\underline{\Psi}$ normal to the line-of-sight. The basic information which can be obtained from such a device, the basic limitations present, and a method of circumventing these limitations are presented in the next section.

IV. SINGLE-BODY TRACKING

In order to derive the equations associated with single-body tracking, a new coordinate system must be defined. This is the uvw, or tracker coordinate system, shown with the epq system in Figure 7. The v axis is defined as being coincident with the line-of-sight to the body at an altitude angle η from the eq plane, u lies in the eq plane and w completes the right-handed orthogonal set. Also shown are the line-of-sight to the body being tracked and the e'pq' system, where q' is defined as being coincident with u at the time the body is first acquired by the tracker. The e'pq' and epq systems remain fixed relative to each other and to the platform or xyz system. The relation between the epq and xyz systems is shown in Figure 5.

First, the equations giving the angular information obtained in tracking a single star will be derived for a vehicle at rest on the earth's surface and for constant gyro drift rates. Then it will be shown that sun-tracking gives essentially no additional information and that small vehicle velocities can be handled easily. Equations for variable drift rates will also be derived.

The line-of-sight to a star remains fixed in inertial space for all practical purposes. Since the epq system has an angular velocity $\underline{\Omega}$ (with respect to inertial space) about p, the uvw system therefore has an angular velocity $-\underline{\Omega}$ (with respect to epq) about p. It is apparent that only components of $\underline{\Psi}$ normal to v can be detected by the tracker. These will be taken as Ψ_u and Ψ_w. (In the actual tracker mechanization, different components will probably be measured. These can be resolved to Ψ_u and Ψ_w,

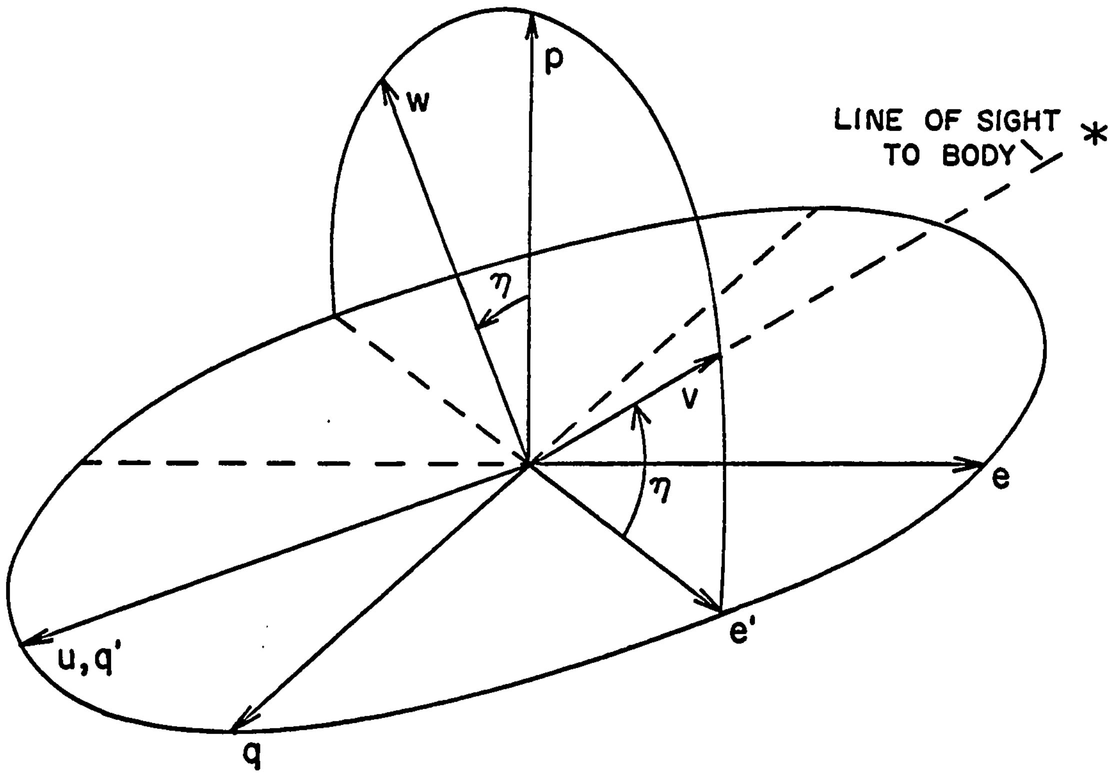

Figure 7. Tracker coordinate system
and line-of-sight to body

however, by a single rotation about v.) Equations 73, 74 and 75 give Ψ_e, Ψ_q and Ψ_p. Examination of equations 42 and 45 shows that $\Psi_{e'}$, $\Psi_{q'}$ and Ψ_p have identical form, with e' and q' replacing e and q respectively. By inspection of Figure 7 and consideration of the rotation of uvw about p at a rate $-\underline{\Omega}$, $\Psi_u(t)$ and $\Psi_w(t)$ can be written

$$\Psi_u(t) = \Psi_{q'}(t) \cos \Omega t - \Psi_{e'}(t) \sin \Omega t \tag{78}$$

$$\begin{aligned} \Psi_w(t) = \Psi_p(t) \cos \eta &- \sin \eta (\Psi_{e'}(t) \cos \Omega t \\ &+ \Psi_{q'}(t) \sin \Omega t) \, . \end{aligned} \tag{79}$$

Substitution of equations 73, 74 and 75 into 78 and 79 (with e and q replaced by e' and q') and use of appropriate trigonometric identities results in

$$\Psi_u(t) = \Psi_{q'o} + \varepsilon_{q'} \frac{\sin \Omega t}{\Omega} + \frac{\varepsilon_{e'}}{\Omega} (\cos \Omega t - 1) \tag{80}$$

$$\begin{aligned} \Psi_w(t) = \Psi_{po} \cos \eta &- \Psi_{e'o} \sin \eta \\ &+ \varepsilon_p (\cos \eta) t - \varepsilon_{e'} \sin \eta \frac{\sin \Omega t}{\Omega} \\ &\frac{\varepsilon_{q'} \sin \eta}{\Omega} (\cos \Omega t - 1) \, , \end{aligned} \tag{81}$$

where the time origin is taken at the time that the star is first acquired. Equations 80 and 81 show that single-star tracking has a very fundamental limitation for the system mechanization of Section II; that only two components of the initial value of $\underline{\Psi}$ can be measured: $\Psi_{q'o}$ and $(\Psi_{po} \cos \eta - \Psi_{e'o} \sin \eta)$, which may be written Ψ_{wo}. The component of $\underline{\Psi}_o$ lying along the initial v axis, $(\Psi_{e'o} \cos \eta + \Psi_{po} \sin \eta)$, is seen to remain coincident with the v axis, and therefore cannot be measured. It might be suspected that use of another coordinate system with different gyro torquing rates could allow the eventual determination of Ψ_{wo}. That this is not so, however, can be seen quite clearly from equation 29, $\left(\frac{d\underline{\Psi}}{dt}\right)_I = \underline{\varepsilon}$. This equation, independent of coordinate systems, indicates that the initial value of $\underline{\Psi}$ remains inertially fixed. Since the line-of-sight to a star is also inertially fixed, the component of $\underline{\Psi}_o$ along the line-of-sight will remain there. This equation also indicates that vehicle motion over the earth, which results, in general, in variable gyro torquing rates, has no effect on $\underline{\Psi}_o$ either. The only method of circumventing this limitation, then, is to alter the basic gyro torquing scheme which gives rise to equation 29. Care must be taken here to preserve the dynamic exactness of the system, however, or the system will not operate accurately in a dynamic environment. This suggests use of the velocity difference signals, equations 77, in gyro torquing, as was discussed briefly in Section III. Before getting into this, however, sun-tracking will be examined.

Due to the earth's orbital motion about the sun, the line-of-sight from the earth to the sun rotates at an angular velocity of about 0.04 degrees/hour with respect to an inertially fixed coordinate system. The angle η also varies, oscillating approximately sinusoidally between limits of $\pm 23\frac{1}{2}$ degrees with a period of one year. Because of these two effects, the angular velocity of uvw relative to epq is not $-\underline{\Omega}$ about p, but is slightly different. In theory, then, all components of $\underline{\Psi}_o$ could be detected; it will be seen, however, that this is impractical.

In order to study sun-tracking, equations 78 and 79 are rewritten with Ω replaced by $\Omega-\delta$:

$$\Psi_u(t) = \Psi_{q'}(t)\cos(\Omega-\delta)t - \Psi_{e'}(t)\sin(\Omega-\delta)t \qquad (82)$$

$$\Psi_w(t) = \Psi_p(t) \cos \eta - \sin \eta(\Psi_{e'}(t) \cos (\Omega-\delta)t$$
$$+ \Psi_{q'}(t) \sin (\Omega-\delta)t) \quad , \tag{83}$$

where $\delta = 0.04$ degrees/hour = 0.0007 radians/hour. A "worst-case" analysis will be made in the case of variations in η, by assuming that it is changing at its maximum rate. Since η can be written

$$\eta \doteq 0.41 \sin \frac{2\pi t}{365(24)} \quad , \tag{84}$$

with η in radians and t in hours, the maximum rate occurs when $\eta = 0$ and is

$$\dot{\eta}_{max} = 0.41 \left(\frac{2\pi}{365(24)}\right) \text{ radians/hour}$$
$$= 2.9\text{x}10^{-4} \text{ radians/hour} . \tag{85}$$

Since $\dot{\eta}_{max}$ and δ are so small, first-order approximations will be made for sines and cosines involving these quantities; that is, for example, $\cos(\Omega-\delta)t = \cos \Omega t \cos \delta t + \sin \Omega t \sin \delta t \doteq \cos \Omega t + \delta t \sin \Omega t$. Equations 82 and 83 can then be written

$$\Psi_u(t) = \Psi_{q'}(t) \cos \Omega t - \Psi_{e'}(t) \sin \Omega t$$
$$+ \delta t(\Psi_{q'}(t) \sin \Omega t + \Psi_{e'}(t) \cos \Omega t) \tag{86}$$

$$\Psi_w(t) = \Psi_p(t) - \dot{\eta}t(\Psi_{e'}(t) \cos \Omega t$$
$$+ \Psi_{q'}(t) \sin \Omega t)$$
$$-\dot{\eta}\delta t^2(\Psi_{e'}(t) \sin \Omega t - \Psi_{q'}(t) \cos \Omega t) \quad , \tag{87}$$

where the nominal value of η is zero, consistent with the maximum rate used.

Substitution of equations 73, 74 and 75 into these equations yields

$$\Psi_u(t) = \Psi_{q'o} + \varepsilon_{q'} \frac{\sin \Omega t}{\Omega} + \frac{\varepsilon_{e'}}{\Omega} (\cos \Omega t-1)$$

$$+ \delta t \left(\Psi_{e'o} + \varepsilon_{e'} \frac{\sin \Omega t}{\Omega} + \frac{\varepsilon_{q'}}{\Omega} (\cos \Omega t-1)\right) \tag{88}$$

$$\Psi_w(t) = \Psi_{po} + \varepsilon_p t - \dot{\eta} t \, (\Psi_{e'o} + \varepsilon_{e'} \frac{\sin \Omega t}{\Omega}$$

$$+ \frac{\varepsilon_{q'}}{\Omega} (\cos \Omega t - 1))$$

$$+ \dot{\eta} \delta t^2 \, (\Psi_{q'o} + \varepsilon_{q'} \frac{\sin \Omega t}{\Omega} + \frac{\varepsilon_{e'}}{\Omega} (\cos \Omega t - 1)) \tag{89}$$

It is now convenient to use small-angle approximations for sin Ωt and cos Ωt in order to see the effects of δ and $\dot{\eta}$ more clearly. Thus sin Ωt is replaced by Ωt and cos Ωt by $1 - \frac{\Omega^2 t^2}{2}$, giving

$$\Psi_u(t) = \Psi_{q'o} + (\varepsilon_{q'} + \delta \Psi_{e'o}) t$$

$$+ \varepsilon_{e'} \left(- \frac{\Omega}{2} + \delta\right) t^2 + \varepsilon_{q'} \left(\frac{\delta \Omega}{2}\right) t^3 \tag{90}$$

$$\Psi_w(t) = \Psi_{po} + (\varepsilon_p - \dot{\eta} \Psi_{e'o}) t$$

$$- \dot{\eta} (\varepsilon_{e'} - \delta \Psi_{q'o}) t^2$$

$$+ \dot{\eta} \left(\frac{\varepsilon_{q'} \Omega}{2} + \delta \varepsilon_{q'}\right) t^3$$

$$- \dot{\eta} \delta \frac{\varepsilon_{e'} \Omega}{2} t^4 \quad . \tag{91}$$

It will be seen later that the small-angle approximations are valid in that required observation time is short compared to 24 hours.

For comparison with equations 90 and 91, equations 80 and 81 are now rewritten using the same small-angle approximations and for $\eta = 0$:

$$\Psi_u(t) = \Psi_{q'o} + \varepsilon_{q'} t - \frac{\Omega}{2} \varepsilon_{e'} t^2 \tag{92}$$

$$\Psi_w(t) = \Psi_{po} + \varepsilon_p t \quad . \tag{93}$$

Comparison of equation 92 with 90 shows that the linear term has an additional term, $\delta\Psi_{e'o}$, and that the quadratic term is modified by $\delta\varepsilon_{e'}$. It will be assumed here that $\Psi_{e'o}$ = 10 minutes of arc, to evaluate the effect of the additional linear term. (This corresponds to a position error of about 10 miles, which is quite large.) The additional linear term then has a magnitude of about 0.0001 degrees/hour, and the determination of $\varepsilon_{q'}$ would be in error by that amount. As will be seen, an error of this magnitude is certainly tolearable, at least in the numerical example used later. In the quadratic term of equation 90, it can be seen that the determination of $\varepsilon_{e'}$ would be in error by a factor $\frac{2\delta}{\Omega}$ or about 0.005. Thus, $\varepsilon_{e'}$ could be determined to 0.5%, which is also quite acceptable. The third-order coefficient in equation 90 has a magnitude, for $\varepsilon_{q'}$ of 0.01 degrees/hour, of about 7x10^{-7} degrees/hour^3. Thus, for a two-hour observation time, this term would reach a maximum value of 0.02 seconds or arc, which is certainly negligible. Since $\dot{\eta} < \delta$, the additional terms in equation 91 are also certainly negligible. There is an additional fourth-order term, with coefficient $\frac{\dot{\eta}\delta\ \varepsilon_{e'}\Omega}{2}$. Again assuming $\varepsilon_{e'}$ = 0.01 degrees/hour, this term has a magnitude of about 10^{-9} degrees/hour^4, which gives a maximum value for a two-hour observation time of about 6x10^{-5} seconds of arc, which is completely negligible. These numerical examples also illustrate that, although in theory equations 90 and 91 allow determination of $\underline{\Psi}_o$ completely, such a procedure would be clearly impractical.

Thus sun-tracking does not, for all practical purposes, circumvent the fundamental limitation discussed earlier for star-tracking. The numerical examples given, in fact, show that equations 80 and 81, or their small angle equivalents, can be used with satisfactory accuracy in determining the available components of $\underline{\Psi}$ by sun-tracking.

The foregoing discussion is not meant to imply, however, that the direction cosines of the line-of-sight to the sun can be computed in such a simple manner. It can be shown that such a procedure would lead to unacceptably large errors in the determination of $\underline{\Psi}$; thus, these direction cosines must be computed quite accurately. This can be done relatively easily, and is discussed in Section II.

It was pointed out previously that vehicle motion over the earth does nothing to circumvent the basic limitation in determing $\underline{\Psi}_o$. Vehicle motion will, however, result in changes in equations 80 and 81, because the angular velocity of xyz, and hence that of epq, changes. This effect may be examined by use of equation 28, which indicates that $\underline{\Omega}$, which has been used thus far, must be replaced by $\underline{\omega}_c = \underline{\Omega} + \underline{\rho}$. The components of $\underline{\omega}$, the angular velocity of xyz relative to an earth-fixed set, are given by equations 11 and 17. It will be assumed (as was done earlier) that the ship is located at a latitude of 45°. Then, from equations 11 and 17,

$$\rho_z = \rho_y = \frac{V_x}{R} \tag{94}$$

$$\rho_x = -\frac{V_y}{R} \quad . \tag{95}$$

Thus, the vector $\underline{\omega_c}$ can be seen to be made up of $\underline{\Omega}$, constant and along p, and $\underline{\rho}$, which lies in the ep plane, by equation 94. A reasonable upper bound on ship velocity is 30 knots (about 50 feet/second). Then, assuming $V_x = V_y = 35$ feet/sec (a northeast course),

$$\rho_z = \rho_y = 0.3\ ^\circ/\text{hour} \tag{96}$$

$$\rho_x = -0.3^\circ/\text{hour} \tag{97}$$

The vector $\underline{\omega_c}$ will then differ from $\underline{\Omega}$ in magnitude by about 2%, and in direction by about 0.02 radian. A new coordinate system, similar to epq, could be constructed with one axis coincident with $\underline{\omega_c}$. Solutions for the components of $\underline{\Psi}$ in this system would then be identical in form to equations 73, 74 and 75, with different subscripts and with Ω replaced by ω_c. If the components of $\underline{\Psi}$ detected by the tracker are resolved to new uw axes (denoted by u' w') with the same relation to the new coordinate system as u and w have to epq, the equations for $\Psi_{u'}$ and $\Psi_{w'}$ would then have the same form as equations 78 and 79, again with Ω replaced by ω_c and with new subscripts. It then follows that equations identical in form to equations 80 and 81 would result, and the problem thus remains the same. Variable velocity during tracking complicates the problem considerably; it will therefore be assumed that a constant velocity is maintained. This does not seem to be an unduly restrictive requirement, as a mechanization such as this is useful primarily in mid-ocean areas where constant velocities can be easily maintained. With this assumption then, it is convenient to use $\underline{\omega_c} = \underline{\Omega}$, because the work thus far has been based on this value. As was pointed out, the general solution using $\underline{\omega_c}$ is identical in form to that obtained using $\underline{\Omega}$; further, for a 30-knot velocity (which most ships cannot sustain), $\underline{\omega_c}$ differs from $\underline{\Omega}$ by only about 2% in magnitude and about 0.02 radian in direction.

The equations for Ψ_u and Ψ_w in their most general form will now be derived. These equations take into consideration variable (or random) gyro drift rates, and are derived from equations 60. First it is convenient to rewrite equations 60 (using expansions for cos Ω(t-u) and sin Ω(t-u) and the e'pq' system) as

$$\begin{aligned}\Psi_{e'}(t) &= \Psi_{e'o} \cos \Omega t - \Psi_{q'o} \sin \Omega t \\ &+ \cos \Omega t \int_o^t [\varepsilon_{e'}(u) \cos \Omega u + \varepsilon_{q'}(u) \sin \Omega u] du \\ &+ \sin \Omega t \int_o^t [\varepsilon_{e'}(u) \sin \Omega u - \varepsilon_{q'}(u) \cos \Omega u] du \end{aligned} \tag{98}$$

$$\begin{aligned}\Psi_{q'}(t) &= \Psi_{q'o} \cos \Omega t + \Psi_{e'o} \sin \Omega t \\ &+ \cos \Omega t \int_o^t [\varepsilon_{q'}(u) \cos \Omega u - \varepsilon_{e'}(u) \sin \Omega u] du \\ &+ \sin \Omega t \int_o^t [\varepsilon_{q'}(u) \sin \Omega u + \varepsilon_{e'}(u) \cos \Omega u] du \end{aligned} \tag{99}$$

$$\Psi_p(t) = \Psi_{po} + \int_o^t \varepsilon_p(u) du \tag{100}$$

Substitution of these equations into equations 78 and 79 yields

$$\Psi_u(t) = \Psi_{q'o} + \int_o^t [\varepsilon_{q'}(u) \cos \Omega u - \varepsilon_{e'}(u) \sin \Omega u] du \tag{101}$$

$$\begin{aligned}\Psi_w(t) &= (\Psi_{po} \cos \eta - \Psi_{e'o} \sin \eta) \\ &+ \cos \eta \int_o^t \varepsilon_p(u) du \\ &- \sin \eta \int_o^t [\varepsilon_{e'}(u) \cos \Omega u + \varepsilon_{q'}(u) \sin \Omega u] du \quad . \end{aligned} \tag{102}$$

These are the most general forms for Ψ_u and Ψ_w. Again, as expected, only two components of $\underline{\Psi}_o$ can be obtained.

As was discussed earlier in this section, the only method of circumventing this limitation is to alter the basic gyro torquing scheme. A "natural" and dynamically exact way of doing this is to utilize velocity difference signals, as was done in damping the 84-minute modes. Here, however, it is not necessarily desirable to damp the 24-hour modes; what is desired is to alter the normal modes in such a manner as to make all three components of $\underline{\Psi}_o$ observable by the tracking process, and within a relatively short time (since the sun is available for about 12 hours per day). In other words, it is desired to move the oscillatory poles of equation 58 as far from $\pm j\Omega$ as is feasible. There are many possible schemes which come to mind. One of the simplest of such schemes is the one selected, and is derived in the following paragraphs.

The derivation begins with the y-component of equation 48:

$$V_{dy} = \delta\dot{R}_y - \delta V_{ry} \quad . \tag{103}$$

This signal is available within the system computer. The quantity $\delta\dot{R}_y$ is found implicitly by differentiating the second of equations 51, assuming V_{ry} and ∇_y are constants:

$$\delta\dddot{R}_y + K_1\delta\ddot{R}_y + \omega_o^{\;2}\delta\dot{R}_y = g\dot{\Psi}_x = g\dot{\Psi}_e \;, \tag{104}$$

since the e and x axes are coincident. The assumption that ∇_y and δV_{ry} are constants is made to simplify subsequent work, and because the major components of these quantities are constant. The main contributor to $\delta\underline{V}_r$ is ocean current, which, in most areas, remains constant over an area consistent with the operating period; accelerometer biases can be expected to remain approximately constant also over such time periods. The differential operation of equation 104 $(D^2 + K_1D + \omega_o^{\;2})$, applied to equation 103, and use of equation 104 gives

$$\ddot{V}_{dy} + K_1\dot{V}_{dy} + \omega_o^{\,2} V_{dy}$$

$$= g\dot{\Psi}_e - \omega_o^{\,2}\, \delta V_{ry} \;, \tag{105}$$

where, again, the assumption is made that δV_{ry} is constant. This operation is thus seen to give a quantity containing $\dot{\Psi}_e$, which will be useful in altering the normal modes of $\underline{\Psi}$ propagation. It is now necessary to examine equations 45, which determine the normal modes of $\underline{\Psi}$ propagation. The system determinant is given in equations 58 in Laplace transform notation, with poles at $s = \pm j\Omega$. One method of altering the normal modes (or moving the poles) would be to introduce a quantity proportional to equation 105 as an additional torquing rate to the q-gyro. (This can be accomplished by an appropriate resolution onto the y and z gyros of the actual system.) Such a scheme, however, gives a system determinant of $s^2 + Ks + \Omega^2$. For small values of K, the poles are very close to $\pm j\Omega$ and nothing has been accomplished; for large K, one pole is very near the origin and the other is on the negative real axis, giving an over-damped system. Such a scheme, therefore, is not very desirable. A better scheme (and the one selected) uses, as an additional torquing rate applied to the q-gyro, a quantity proportional to the integral of equation 105. This additional rate is denoted by $T_q(t)$ and is given by

$$T_q(t) = \frac{K}{g}\int_o^t [\ddot{V}_{dy}(u) + K_1\dot{V}_{dy}(u) + \omega_o^{\,2}\, V_{dy}(u)]du$$

$$= K(\Psi_e(t) - \Psi_{eo}) - \frac{K\omega_o^{\,2}}{g}\,\delta V_{ry} t \;. \tag{106}$$

With this additional rate, equations 45 become

$$\dot{\Psi}_e + \Omega\Psi_q = \varepsilon_e \tag{107}$$

$$\dot{\Psi}_p \qquad = \varepsilon_p \tag{108}$$

and

$$\dot{\Psi}_q - \Omega\Psi_e = \varepsilon_q + K\Psi_e - K\Psi_{eo} - \frac{K\omega_o^{\;2}}{g}\,\delta V_{yr} t \,, \tag{109}$$

or

$$\dot{\Psi}_q - (\Omega+K)\Psi_e = \varepsilon_q - K\Psi_{eo} - \frac{K\omega_o^{\;2}}{g}\;\delta V_{yr} t \,. \tag{110}$$

It is seen that Ψ_p propagates as before. Equations 107 and 110, in Laplace transform notation, are

$$\begin{pmatrix} s & \Omega \\ -(\Omega+K) & s \end{pmatrix}\begin{pmatrix} \Psi_e(s) \\ \Psi_q(s) \end{pmatrix} = \begin{pmatrix} \varepsilon_e(s) + \Psi_{eo} \\ \varepsilon_q(s) - \dfrac{K\Psi_{eo}}{s} + \Psi_{qo} - \dfrac{K\omega_o^{\;2}}{g}\,\dfrac{\delta V_{yr}}{s^2} \end{pmatrix} \tag{111}$$

The poles for this mechanization are located at $s = \pm j\sqrt{\Omega(\Omega+K)}$, and thus, by selecting an appropriate value of K, can be moved out along the imaginary axis as far as desired. It appears, then, that this mechanization should allow determination of $\underline{\Psi}_o$ completely in a reasonable time. This is shown to be true in what follows.

First, it is convenient to denote the natural frequency of the normal modes by

$$\omega_1^{\;2} = \Omega(\Omega+K) \,, \tag{112}$$

and let

$$\Delta V' = -\frac{K\omega_o^{\;2}}{g}\,\delta V_{yr} \,. \tag{113}$$

Solutions of equation 111 can then be written

$$\Psi_e(s) = \frac{1}{s^2 + \omega_1^{\;2}}\left[s(\varepsilon_e(s) + \Psi_{eo}) - \Omega(\varepsilon_q(s) + \frac{\Delta V'}{s^2} - \frac{K\Psi_{eo}}{s} + \Psi_{qo})\right] \tag{114}$$

$$\Psi_q(s) = \frac{1}{s^2 + \omega_1^2} [s(\varepsilon_q(s) - \frac{K\Psi_{eo}}{s} + \Psi_{qo} + \frac{\Delta V'}{s^2})$$

$$+ (\Omega+K) (\varepsilon_e(s) + \Psi_{eo})] , \tag{115}$$

and, in the time domain,

$$\Psi_e(t) = \Psi_{eo}[\cos \omega_1 t + \frac{K\Omega}{\omega_1 2} (1 - \cos \omega_1 t)]$$

$$- \frac{\Omega}{\omega_1} \Psi_{qo} \sin \omega_1 t - \frac{\Omega}{\omega_1 3} \Delta V'(\omega_1 t - \sin \omega_1 t)$$

$$+ \int_o^t [\varepsilon_e(u) \cos \omega_1(t-u) - \frac{\Omega}{\omega_1} \varepsilon_q(u) \sin \omega_1(t-u)]du \tag{116}$$

$$\Psi_q(t) = \Psi_{qo} \cos \omega_1 t + \frac{\Omega}{\omega_1} \Psi_{eo} \sin \omega_1 t$$

$$+ \frac{\Delta V'}{\omega_1 2} (1 - \cos \omega_1 t)$$

$$+ \int_o^t [\varepsilon_q(u) \cos \omega_1(t-u) + \frac{(\Omega+K)}{\omega_1} \varepsilon_e(u) \sin \omega_1(t-u)]du \tag{117}$$

The equation for Ψ_p remains unchanged:

$$\Psi_p(t) = \Psi_{po} + \int_o^t \varepsilon_p(u)du . \tag{118}$$

At this point, it was decided to assume that equations 80 and 81 (or 101 and 102) would be used to find the three gyro drift rates and two components of $\underline{\Psi}_o$ while operating in the normal gyro torquing mode, and that these quantities could be determined and corrected to such accuracy that they can be neglected in equations 116, 117 and 118. Section VII shows that, for the numerical examples chosen, this assumption is valid. The decision to operate in two "phases" has the merit that the solution for all components of $\underline{\Psi}$ from one set of complex equations (116, 117 and 118) is replaced by the solution for $\underline{\Psi}$ from two sets of simpler equations (101, 102 and equations to be derived shortly). A further simplifying assumption is made; that operation in the new or "decoupled" gyro torquing mode commences at local apparent noon (when v lies in the pq plane). This assumption also simplifies the work which follows, and should give a reasonable evaluation of system performance, in that actual operation in this mode should commence at about noon.

The latter assumption implies that, at the commencement of operation in the decoupled mode, $\underline{\Psi}_o$ lies entirely along the v axis and thus in the pq plane. Then, by this and the first assumption,

$$\Psi_{eo} = \varepsilon_e = \varepsilon_p = \varepsilon_q = 0 \tag{119}$$

$$\Psi_{qo} = \Psi_{od} \cos \eta \tag{120}$$

$$\Psi_{po} = \Psi_{od} \sin \eta \quad . \tag{121}$$

Equations 116, 117 and 118 then become

$$\Psi_e(t) = -\frac{\Omega}{\omega_1} \Psi_{od} \cos \eta \sin \omega_1 t - \frac{\Omega}{\omega_1^{\,3}} \Delta V'(\omega_1 t - \sin \omega_1 t) \tag{122}$$

$$\Psi_q(t) = \Psi_{od} \cos \eta \cos \omega_1 t + \frac{\Delta V'}{\omega_1^{\,2}} (1 - \cos \omega_1 t) \tag{123}$$

$$\Psi_p\,(t) = \Psi_{od} \sin \eta \quad . \tag{124}$$

Reference to Figure 7 shows that Ψ_u and Ψ_w can be written

$$\Psi_u(t) = -\Psi_e(t) \cos \Omega t - \Psi_q(t) \sin \Omega t \tag{125}$$

$$\Psi_w(t) = \Psi_p(t) \cos \eta$$

$$+ \sin \eta(\Psi_e(t) \sin \Omega t - \Psi_q(t) \cos \Omega t) \quad . \tag{126}$$

Substitution of equations 122, 123 and 124 into 125 and 126 then yields

$$\Psi_u(t) = \Psi_{od} \cos \eta \left[\frac{\Omega}{\omega_1} \sin \omega_1 t \cos \Omega t\right.$$

$$\left. - \cos \omega_1 t \sin \Omega t\right]$$

$$+ \frac{\Delta V'}{\omega_1^{\,2}} \left[\frac{\Omega}{\omega_1} (\omega_1 t - \sin \omega_1 t) \cos \Omega t\right.$$

$$\left. - (1 - \cos \omega_1 t) \sin \Omega t\right] \tag{127}$$

$$\Psi_w(t) = \sin \eta\{\Psi_{od} \cos \eta \left[1 - \cos \omega_1 t \cos \Omega t\right.$$

$$\left. - \frac{\Omega}{\omega_1} \sin \omega_1 t \sin \Omega t\right]$$

$$- \frac{\Delta V'}{\omega_1^{\,2}} \left[(1 - \cos \omega_1 t) \cos \Omega t\right.$$

$$\left. + \frac{\Omega}{\omega_1} (\omega_1 t - \sin \omega_1 t) \sin \Omega t\right]\} \quad . \tag{128}$$

The reasons for the simplifying assumptions made earlier now are more obvious; equations 127 and 128 are still reasonably complex. These equations do, however, allow the determination of Ψ_o (and $\Delta V'$, which must also be found, since it is now a driving function in $\underline{\Psi}$). It is, in fact, possible to determine Ψ_o and $\Delta V'$ from either of these two equations, since the time functions multiplying Ψ_o and $\Delta V'$ are different in form. This possibility will be discussed later. It is interesting to note in passing that equation 122 illustrates one of the undesirable aspects of using velocity differences in gyro torquing; namely that $\Delta V'$ gives rise to term linear with time in Ψ_e. This behavior is typical of such systems, and indicates why they are not used during navigation modes of operation.

In summary, this section presents a derivation of the basic information which can be obtained using a sun-tracker with conventional gyro torquing (equations 101 and 102), discusses the inability of this mode of operation to determine $\underline{\Psi}_o$ completely, and presents a gyro torquing mechanization utilizing velocity differences (equation 106) which allows determination of the remaining component of $\underline{\Psi}_o$ (equation 127 or 128). The remaining problems to be solved are the determination of methods for extracting the desired quantities from equations 101, 102 and 127 or 128, and the mechanization of an error-correction operation. It must be recognized at this point that the sun-tracker will introduce errors into Ψ_u and Ψ_w. The observed values of these quantities will thus be denoted as Ψ_{ub} and Ψ_{wb}, where

$$\Psi_{ub}(t) = \Psi_u(t) + n_u(t) \tag{129}$$

$$\Psi_{wb}(t) = \Psi_w(t) + n_w(t) \quad . \tag{130}$$

The functions $n_u(t)$ and $n_w(t)$ represent tracker errors or noise in the u and w channels. These quantities will be considered random in nature; thus, the determination of $\underline{\Psi}$ becomes a statistical filter problem. General statistical filter optimization theory is discussed in the next section, and the optimum filters for this application are derived there.

V. DERIVATION OF OPTIMUM FILTER EQUATIONS

In order to derive the equations specifying the optimum filters to be used in determing $\underline{\Psi}$, certain concepts and definitions must first be established. Frequent reference will be made to stationary and non-stationary random processes. A random process is defined here as a collection of functions of time, each of which is non-deterministic or unpredictable except in a statistical sense. All sample functions of such a process have the same statistical properties, which are generally included in the definition of a random process. A stationary (or time-stationary) random process is defined as a random process with statistical properties that are independent of the time origin; on the other hand, the statistical properties of a non-stationary random process do (in general) depend on the time origin. An often-quoted example of a stationary random process is the thermal noise voltage generated within a resistor held at a constant temperature. The noise voltage existing during a particular time period, then, is an example of a sample function of this process. The statistical properties of this voltage certainly would not be expected to depend on the time origin; thus the process is said to be stationary. The random-walk example of probability theory is a non-stationary random process; here the variance of the "walker's" distance from the origin can be shown to be dependent on the time elapsed since the beginning of the experiment. For more detailed discussions of these concepts, the reader is referred to Brown and Nilsson (4), Davenport and Root (5), or Laning and Battin (6). In this study, it will be assumed that the sun-tracker errors, $n_u(t)$ and $n_w(t)$, are sample functions from a stationary random process.

Gyro drift rates, $\varepsilon_e(t)$, $\varepsilon_p(t)$ and $\varepsilon_q(t)$ will be assumed to be made up of constant (but unkown) and stationary random components. That is, for example,

$$\varepsilon_e(t) = \varepsilon_{ec} + \varepsilon_{er}(t) \ , \tag{131}$$

where ε_{ec} is a constant but unknown quantity and $\varepsilon_{er}(t)$ is a sample function from a stationary random process. Such a model has been found from experience to characterize gyro drift rate adequately. All components of $\underline{\Psi}_o$ are, of course, constants but again unknown; $\Delta V'$ is also assumed constant but unknown, as was discussed earlier. These constant but unknown quantities will be regarded as random variables, as they can reasonably be expected to vary in a random or unpredictable manner from day to day. The statistical properties of all of these quantities are assumed to be known; it will be seen in what follows that only second-order statistical properties are required (i.e., mean values and covariances). A further simplification is achieved by the assumption that the mean values of all random variables considered here are zero. This is a very reasonable assumption; for example, the average or mean value of $\Delta V'$, proportional to north reference velocity error, when taken over all time and all possible ship locations will certainly be extremely small if not identically zero. Likewise, gyros can be expected to drift in one direction just as often as in the opposite direction. In fact, if any of these quantities did have mean values, corrections would logically be made for them, and the resultant processes would then have zero mean values.

The basic problem can now be stated: given $\Psi_{ub}(t)$ and $\Psi_{wb}(t)$ from equations 129 and 130 (where $\Psi_u(t)$ and $\Psi_w(t)$ are given by equations 101 and 102 for the normal operating mode, and by equations 127 and 128 for the decoupled mode), find the linear filters which give the least mean-square errors in estimating the values of $\Delta V'$ and components of $\underline{\Psi}$ for all times after initiation of the filtering operations. An example will serve to clarify the above statement; an estimate of $\Psi_{q'o}$ when operating in the normal mode can be written

$$\Psi_{q'oE}(t) = \int_0^t (w_{11}(t,\tau)\Psi_{ub}(\tau) + w_{12}(t,\tau)\Psi_{wb}(\tau))d\tau \qquad (132)$$

where $w_{11}(t,\tau)$ and $w_{12}(t,\tau)$ are the weighting functions of the linear filters which operate on $\Psi_{ub}(t)$ and $\Psi_{wb}(t)$, respectively. The basic problem, then, is to find the weighting functions $w_{11}(t,\tau)$ and $w_{12}(t,\tau)$ which minimize, for all $t > 0$, the expected value of $(\Psi_{q'oE}(t) - \Psi_{q'o})^2$. This expected value will be written $\overline{(\Psi_{q'oE}(t) - \Psi_{q'o})^2}$, and is not a time average in the normal sense, but rather an ensemble average, or average over all possible values of the quantity which could exist on a day-to-day basis. The superscript "bar" will be used in general to denote such an average.

The choice of a mean-square error criterion of "goodness" is somewhat arbitrary; it is a common and traditional choice in similar problems, however, because only second-order statistics are required. The restriction of the choice of filters to linear ones is also arbitrary. This restriction is made because no general solution to the nonlinear filter problem is known. In fact, a general solution to the linear filter problem (giving the filter weighting function explicitly) is not known. It should be noted that the linear filters here will be chosen from the most general class of such filters; thus the weighting functions in equation 132 are written for time-varying linear filters.

The first solution to such an optimum filter problem which specifies the functional form of the optimum weighting function is due to Kolmogoroff and Wiener. This theory is now regarded as "classical" and may be found in a number of references (for example, Brown and Nilsson (4), Davenport and Root (5) or Laning and Battin (6)). The assumptions made in the Wiener-Kolmogoroff theory are quite restrictive, however; it is assumed that both the signal and the noise are sample functions from stationary random processes and that the filter operates in a steady-state condition (that the infinite past history of signal and noise are available). It follows, then, that the filter is of the linear, constant-coefficient type, since the time origin is immaterial. The problem being considered here does not fall into this category for two reasons; the observation time is finite, and the signal is non-stationary (as will be seen later). Thus Wiener-Kolmogoroff theory cannot be applied.

In 1952, Booton (7) extended the Wiener-Kolmogoroff theory to include non-stationary inputs and finite observation time, and derived the integral equation which the optimum filter weighting function must satisfy. Booton, however, considered the case where only one signal (with additive noise) is available; from equations 129 and 130 it can be seen that, in this study, two signals are available. The integral equations which the optimum weighting functions must satisfy will therefore be derived here in a general form. The derivation is similar to Booton's, and utilizes the calculus of variations, as might be expected.

First, it is assumed that the quantity to be estimated is $s_i(t)$. Thus $s_i(t)$ represents $\Psi_{q'o}$, $\varepsilon_{q'}(t)$, $\varepsilon_{e'}(t)$, $\Psi_{po} \cos \eta - \Psi_{e'o} \sin \eta$, $\varepsilon_p(t)$, Ψ_{od} or $\Delta V'$, and the equations will be generally applicable. The estimate of $s_i(t)$ can be written, as in equation 132, as

$$s_{iE}(t) = \int_o^t [w_{i1}(t,\tau)\Psi_{ub}(\tau) + w_{i2}(t,\tau)\Psi_{wb}(\tau)]d\tau \qquad (133)$$

The estimation error is then

$$e_{si}(t) = s_{iE}(t) - s_i(t) , \qquad (134)$$

and the mean-square estimation error is

$$\overline{e_{si}^2(t)} = \overline{(s_{iE}(t) - s_i(t))^2}$$

$$= \overline{s_{iE}^2(t)} + \overline{s_i^2(t)} - 2\,\overline{s_{iE}(t)s_i(t)} , \qquad (135)$$

since the averaging process (an integration) is certainly distributive with respect to addition. Equation 135 is now rewritten, using equation 133, to yield

$$\overline{e_{si}^2(t)} = \overline{s_i^2(t)}$$

$$+\overline{[\int_o^t [w_{i1}(t,\tau)\Psi_{ub}(\tau) + w_{i2}(t,\tau)\Psi_{wb}(\tau)]d\tau]^2}$$

$$\overline{-2[s_i(t) \int_o^t [w_{i1}(t,\tau)\Psi_{ub}(\tau) + w_{i2}(t,\tau)\Psi_{wb}(\tau)]d\tau]} , \qquad (136)$$

or

$$\overline{e_{si}^2(t)} = \overline{s_i^{\,2}(t)}$$

$$+ \overline{\int_o^t \int_o^t [w_{i1}(t,\tau)w_{i1}(t,x)\Psi_{ub}(\tau)\Psi_{ub}(x)}$$

$$+ \overline{w_{i1}(t,\tau)w_{i2}(t,x)\Psi_{ub}(\tau)\Psi_{wb}(x)}$$

$$+ \overline{w_{i1}(t,x)w_{i2}(t,\tau)\Psi_{ub}(x)\Psi_{wb}(\tau)}$$

$$+ \overline{w_{i2}(t,\tau)w_{i2}(t,x)\Psi_{ub}(\tau)\Psi_{wb}(x)]d\tau}\, dx$$

$$- 2 \overline{\int_o^t [w_{i1}(t,\tau)s_i(t)\Psi_{ub}(\tau) + w_{i2}(t,\tau)s_i(t)\Psi_{wb}(\tau)]d\tau} \qquad (137)$$

where the square of $s_{iE}(t)$ has been written as a double integral with τ and x as dummy variables of integration, and in the third term of equation 136, $s_i(t)$ has been written within the integral, since integration is with respect to τ. The next step is to interchange the order of averaging and integration. This is equivalent to interchanging the order of integration of two integrals (one improper) with a parameter. For example, in the last term of equation 137, one of the terms can be written

$$\overline{f(t)} = -2 \overline{\int_o^t [w_{i1}(t,\tau)s_i(t)\Psi_{ub}(\tau)]d}$$

$$= \int f(t)\ p(f(t))\ df \quad , \qquad (138)$$

where p(f(t)) is the probability density function of f(t) and the integral is taken over all possible values of f(t). Since the only random variables present are $s_i(t)$ and $\Psi_{ub}(t)$, equation 138 can be written

$$\overline{f(t)} = \iint f(t)\ p(s_i, \Psi_{ub})\ ds_i\ d\Psi_{ub}\ , \tag{139}$$

or

$$\overline{f(t)} = \iint [-2 \int_o^t w_{i1}(t,\tau)s_i(t)\Psi_{ub}(\tau)d\tau]$$

$$p(s_i, \Psi_{ub})ds_i\ d\Psi_{ub} \tag{140}$$

where the double integral is taken over the domains of definition of s_i and Ψ_{ub}. Davenport and Root (5) show that if

$$\int_o^t \overline{[|w_{i1}(t,\tau)s_i(t)\Psi_{ub}(\tau)|]}d\tau < \infty \tag{141}$$

then the averaging process and the integration can be interchanged. It will therefore be assumed that equation 141 in a general form is satisfied by all functions considered here. This assumption is made freely in all known texts and papers on this subject, and seems reasonable in view of the fact that all random processes being considered here are generated by physical mechanisms which are well-behaved.

Interchanging the averaging processes and integrations in equation 137, utilizing the distributive property of the averaging process and combining the second and third terms of the double integral (since the dummy variables x and τ can be interchanged) gives

$$\overline{e_{si}^2(t)} = \overline{s_i^2(t)}$$

$$+ \int_o^t \int_o^t [w_{i1}(t,\tau) w_{i1}(t,x) \overline{\Psi_{ub}(\tau)\Psi_{ub}(x)}$$

$$+ 2\, w_{i1}(t,\tau) w_{i2}(t,x) \overline{\Psi_{ub}(\tau)\Psi_{wb}(x)}$$

$$+ w_{i2}(t,\tau) w_{i2}(t,x) \overline{\Psi_{wb}(\tau)\Psi_{wb}(x)}] d\tau\, dx$$

$$- 2 \int_o^t [w_{i1}(t,\tau) \overline{s_i(t)\Psi_{ub}(\tau)} + w_{i2}(t,\tau) \overline{s_i(t)\Psi_{wb}(\tau)}] d\tau \qquad (142)$$

It is now desired to find the functions w_{i1} and w_{i2} which will minimize this quantity. Normally, one restricts the class of functions to those which are physically realizable; that is, those which are identically zero for $\tau > t$. This restriction certainly must also be made here; however, the values of the weighting functions which are of interest are only those for $0 \leq \tau \leq t$ (and $0 \leq x \leq t$), since the weighting functions will be used only over these intervals. Thus physical realizability is automatically assured.

In order to minimize $\overline{e_{si}^2(t)}$, the calculus of variations will be used. This basic approach is discussed in detail in many of the references already given and in Weinstock (8), and consists of replacing $w_{i1}(t,\tau)$ by $w_{i1o}(t,\tau) + \varepsilon_1 g_1(t,\tau)$ and $w_{i2}(t,\tau)$ with $w_{i2o}(t,\tau) + \varepsilon_2 g_2(t,\tau)$, where w_{i1o} and w_{i2o} are the optimum weighting functions, ε_1 and ε_2 are parameters, and g_1 and g_2 are arbitrary functions. With these substitutions, $\overline{e_{si}^2(t)}$ becomes a function of ε_1 and ε_2, and necessary conditions for $\overline{e_{si}^2(t)}$ to possess a minimum value for the weighting functions w_{i1o} and

w_{i2o} are that

$$\left.\frac{\partial}{\partial \varepsilon_1}\left(\overline{e_{si}^2(t,\varepsilon_1,\varepsilon_2)}\right)\right|_{\varepsilon_1,\varepsilon_2 = 0} = 0 \tag{143}$$

$$\left.\frac{\partial}{\partial \varepsilon_2}\left(\overline{e_{si}^2(t,\varepsilon_1,\varepsilon_2)}\right)\right|_{\varepsilon_1,\varepsilon_2 = 0} = 0 \quad . \tag{144}$$

With the indicated substitutions made, equation 142 becomes

$$\begin{aligned}
\overline{e_{si}^2(t,\varepsilon_1,\varepsilon_2)} &= \overline{e_{sio}^2(t)} \\
&+\varepsilon_1\Bigg\{\int_o^t\int_o^t [(g_1(t,\tau)w_{i1o}(t,x) + g_1(t,x)w_{i1o}(t,\tau))\ \overline{\Psi_{ub}(\tau)\Psi_{ub}(x)} \\
&+ 2\ g_1(t,\tau)w_{i2o}(t,x)\ \overline{\Psi_{ub}(\tau)\Psi_{wb}(x)}]d\tau\ dx \\
&- 2\int_o^t g_1(t,\tau)\ \overline{s_i(t)\Psi_{ub}(\tau)}\ d\tau\Bigg\} \\
&+ \varepsilon_2\Bigg\{\int_o^t\int_o^t [2g_2(t,x)w_{i1o}(t,\tau)\overline{\Psi_{ub}(\tau)\Psi_{wb}(x)} \\
&+ (g_2(t,\tau)w_{i2o}(t,x) + g_2(t,x)w_{i2o}(t,\tau))\overline{\Psi_{wb}(x)\Psi_{wb}(\tau)}]d\tau\ dx \\
&- 2\int_o^t g_2(t,\tau)\ \overline{s_i(t)\Psi_{wb}(\tau)}\ d\tau\Bigg\} \\
&+ \varepsilon_1^2\int_o^t\int_o^t [g_1(t,\tau)g_1(t,x)\ \overline{\Psi_{ub}(\tau)\Psi_{ub}(x)}]d\tau\ dx \\
&+ 2\varepsilon_1\varepsilon_2\int_o^t\int_o^t [g_1(t,\tau)g_2(t,x)\ \overline{\Psi_{ub}(\tau)\Psi_{wb}(x)}]d\tau\ dx \\
&+ \varepsilon_2^2\int_o^t\int_o^t [g_2(t,\tau)g_2(t,x)\ \overline{\Psi_{wb}(\tau)\Psi_{wb}(x)}]d\tau\ dx \quad ,
\end{aligned} \tag{145}$$

where $\overline{e^2_{sio}(t)}$ is the value of $\overline{e^2}$ with the optimum weighting functions and is independent of ε_1 and ε_2. Equations 143 and 144 then give, as necessary conditions for minimizing $\overline{e^2}$,

$$\frac{\partial}{\partial\varepsilon_1}\left(\overline{e^2_{si}(t,\varepsilon_1,\varepsilon_2)}\right)\Bigg|_{\varepsilon_1,\varepsilon_2 = 0} = 0$$

$$= \int_o^t g_1(t,\tau)\left\{\int_o^t [w_{i1o}(t,x)\ \overline{\Psi_{ub}(\tau)\Psi_{ub}(x)}\right.$$

$$+ w_{i2o}(t,x)\ \overline{\Psi_{ub}(\tau)\Psi_{wb}(x)}]dx$$

$$\left. - \overline{s_i(t)\Psi_{ub}(\tau)}\right\} d\tau \quad , \tag{146}$$

and

$$\frac{\partial}{\partial\varepsilon_2}\left(\overline{e^2_{si}(t,\varepsilon_1,\varepsilon_2)}\right)\Bigg|_{\varepsilon_1,\varepsilon_2 = 0} = 0$$

$$= \int_o^t g_2(t,\tau)\left\{\int_o^t [w_{i20}(t,x)\ \overline{\Psi_{wb}(\tau)\Psi_{wb}(x)}\right.$$

$$+ w_{i1o}(t,x)\ \overline{\Psi_{ub}(x)\Psi_{wb}(\tau)}]dx$$

$$\left. - \overline{s_i(t)\ \Psi_{wb}(\tau)}\right\} d\tau \tag{147}$$

where, in both equations, the dummy variables of integration (τ and x) have been interchanged where convenient and the factors of 2 have been dropped. Now, since $g_1(t,\tau)$ and $g_2(t,\tau)$ are arbitrary functions, equations 146 and 147 will be satisfied only if the quantities within the large brackets are identically zero for $0 \leq \tau \leq t$. This gives

$$\int_0^t [w_{i1o}(t,x)\ \overline{\Psi_{ub}(\tau)\Psi_{ub}(x)} + w_{i2o}(t,x)\ \overline{\Psi_{ub}(\tau)\Psi_{wb}(x)}]dx$$

$$= \overline{s_i(t)\Psi_{ub}(\tau)}\ ,\quad 0 \leq \tau \leq t\ , \tag{148}$$

and

$$\int_0^t [w_{i2o}(t,x)\ \overline{\Psi_{wb}(\tau)\Psi_{wb}(x)} + w_{i1o}(t,x)\ \overline{\Psi_{ub}(x)\Psi_{wb}(\tau)}]dx$$

$$= \overline{s_i(t)\Psi_{wb}(\tau)}\ ,\quad 0 \leq \tau \leq t\ . \tag{149}$$

These, then, are necessary conditions for minimizing $\overline{e_{si}^2(t)}$; they can also be shown to be sufficient by use of equation 147. If equations 148 and 149 are substituted into equation 147, the terms with coefficients ε_1 and ε_2 go to zero. The remaining terms are then $\overline{e_{sio}^2(t)}$, which is the mean-square error obtained by using w_{i1o} and w_{i2o}, and the last three terms. Interchanging the averaging and integration process again in these last three terms gives

$$\begin{aligned}
\overline{e_{si}^2(t,\varepsilon_1,\varepsilon_2)} &= \overline{e_{sio}^2(t)} \\
&+ \int_0^t \int_0^t [\overline{\varepsilon_1^2 g_1(t,\tau)g_1(t,x)\Psi_{ub}(\tau)\Psi_{ub}(x)} \\
&+ \overline{2\ \varepsilon_1 g_1(t,\tau)\varepsilon_2 g_2(t,x)\Psi_{ub}(\tau)\Psi_{ub}(x)} \\
&+ \overline{\varepsilon_2^2\ g_2(t,\tau)g_2(t,x)\Psi_{wb}(\tau)\Psi_{wb}(x)}]d\tau\ dx\ ,
\end{aligned} \tag{150}$$

which can be written

$$\overline{e_{si}^2(t,\varepsilon_1,\varepsilon_2)} = \overline{e_{sio}^2(t)}$$
$$+ \overline{\left\{\int_o^t [\varepsilon_1 g_1(t,\tau)\Psi_{ub}(\tau) + \varepsilon_2 g_2(t,\tau)\Psi_{wb}(\tau)]d\tau\right\}^2} \quad . \tag{151}$$

Now, the second term in equation 151 is the mean-square value of a real quantity and is therefore non-negative. Thus, if $w_{i1o}(t,\tau)$ and $w_{i2o}(t,\tau)$ satisfy equations 148 and 149, the mean-square error obtained by using them, $\overline{e_{sio}^2(t)}$, is never larger than the mean-square error obtained by using any other weighting functions (since $\varepsilon_1 g_1(t,\tau)$ and $\varepsilon_2 g_2(t,\tau)$ are arbitrary).

The optimum linear filter weighting functions are thus given by equations 148 and 149. Booton's result can be obtained by setting either Ψ_{ub} or Ψ_{wb} equal identically to zero. For example, if $\Psi_{wb} \equiv 0$, equation 148 becomes

$$\int_o^t [w_{i1o}(t,x)\ \overline{\Psi_{ub}(\tau)\Psi_{ub}(x)}]dx = \overline{s_i(t)\Psi_{ub}(\tau)}\ ,\ 0 \le \tau \le t\ , \tag{152}$$

and equation 149 gives the trivial identity 0=0. This is equivalent to using only Ψ_{ub} to estimate s_i and "ignoring" the other signal. The problem of finding w_{i1o} and w_{i2o} is far from solved, however, because equations 148 and 149 are coupled Fredholm equations of the first kind, for which, according to Chang (9), no general solution is known. Several people have succeeded, however, in obtaining solutions or methods of solution of problems similar to this under various restrictive assumptions. Two of these approaches are discussed in some detail in what follows. Other approaches, which are not applicable here, can be found in Davenport and Root (5), Laning and Battin (6) and Chang (9).

Kalman and Bucy (10) approach the optimal filter problem from the "state-transition" point of view, which utilizes first-order matrix differential equations to characterize linear, time-varying dynamical systems of any order. The basic assumptions made are that the "message" process ($\underline{\Psi}$ in this case) can be generated by a linear dynamical system excited by white noise (or that such a system will give statistics identical to second-order to those of the signal) and that the observed signal consists of the message transformed by a time-varying matrix plus white noise. If gyro drift rates and tracker noise have suitable statistical properties (which they will be assumed to have here), the problem being studied satisfies these assumptions, and this approach could be used. Kalman and Bucy derive a nonlinear matrix differential equation, the solution of which specifies the optimum linear filter in matrix form. What Kalman and Bucy have succeeded in doing, then, is to replace the integral equations (equations 148 and 149 here, which must be solved seven times to find the 14 optimum weighting functions) by a nonlinear matrix differential equation. Their result is thus compact in form; in most cases, however, the optimum filter can be found only by numerical techniques, and thus specified by a table of values or a curve-fitting process. Such an approach has the disadvantages of requiring a large system computer memory for storage of these values, and of lending little insight into the mathematical form of the filter weighting function. Because of the first disadvantage given, it was decided not to use the Kalman-Bucy method of finding the optimum filter, since the filter, when found, could probably not be accommodated in a relatively small, special-purpose computer of the type presently used in such applications.

Another approach to the problem at hand is given by Shinbrot (11), who gives a method of solution of equation 152. Shinbrot's basic assumption is that $\overline{\Psi_{ub}(x)\Psi_{ub}(\tau)}$ and $\overline{s_i(t)\Psi_{ub}(\tau)}$ in equation 152 can be written (or closely approximated) in the form

$$\overline{\Psi_{ub}(x)\Psi_{ub}(\tau)} = \sum_{j=1}^{n} a_j(x)b_j(\tau) \quad , \; x \geq \tau \tag{153}$$

$$\overline{s_i(t)\Psi_{ub}(\tau)} = \sum_{j=1}^{n} c_j(t)b_j(\tau) \quad , \; t \geq \tau \tag{154}$$

He then observes that $\overline{\Psi_{ub}(x)\Psi_{ub}(\tau)}$ is always symmetric; that is,

$$\overline{\Psi_{ub}(x)\Psi_{ub}(\tau)} = \overline{\Psi_{ub}(\tau)\Psi_{ub}(x)} \quad . \tag{155}$$

Thus $\overline{\Psi_{ub}(x)\Psi_{ub}(\tau)}$, for $\tau \geq x$, becomes

$$\overline{\Psi_{ub}(x)\Psi_{ub}(\tau)} = \sum_{j=1}^{n} a_j(\tau)b_j(x) \quad , \; \tau \geq x \quad . \tag{156}$$

Shinbrot then uses the vectors $\underline{A}$, $\underline{B}$ and $\underline{C}$ whose elements are a_j, b_j and c_j, respectively, and, with equations 153, 154 and 156, writes equation 152 as

$$\begin{aligned} \underline{C}(t)\cdot\underline{B}(\tau) &= \underline{A}(\tau)\cdot\int_0^\tau \underline{B}(x)w_{ilo}(t,x)dx \\ &+\underline{B}(\tau)\cdot\int_\tau^t \underline{A}(x)w_{ilo}(t,x)dx \quad . \end{aligned} \tag{157}$$

The next step in Shinbrot's derivation is the assumption that $w_{ilo}(t,x)$ can be written

$$w_{ilo}(t,x) = \underline{G}(t)\cdot\underline{\Gamma}(x) \quad . \tag{158}$$

(In equations 157 and 158 the "dot" denotes the conventional vector dot-product.) He then shows that equation 157 can be separated into two integral equations, each depending on a single variable, which result, in general, in linear, time-varying differential equations for the components of $w_{ilo}(t,x)$. It should be noted at this point that the Kalman-Bucy non-linear differential equation can be reduced to linear, time-varying differential equations and, as might be suspected, these equations are identical to Shinbrot's. Thus, both of these approaches succeed in replacing the integral equation with linear differential equations. The Kalman-Bucy approach is more general in that it considers the problem of more than one observed signal. Shinbrot's method, however, could be extended to cover this situation. In either case, though, numerical solutions would almost certainly be necessary for the problem considered here; thus neither of these approaches will be used.

The method of solution of equations 148 and 149 which was selected relies on an approximation based on observed statistical characteristics of gyro drift rates. It is found from experience that gyros of the type used in marine inertial systems exhibit random drift rates with long correlation times; that is, a typical drift rate autocorrelation function can be closely approximated as

$$\overline{\varepsilon_r(t)\varepsilon_r(t+\tau)} = \sigma_r^{\,2}\, e^{-\alpha|\tau|} \quad , \tag{159}$$

where $\frac{1}{\alpha}$ is of the order of 10 hours or more. Since the observation time is expected to be of the order of one or two hours, it was decided to approximate the random drift rate components during the observation time as constants. This approximation leads to a relatively simple solution of equations 148 and 149; the effect of the approximation is shown in a succeeding section. It is also assumed that the tracker noise or error functions,

$n_u(t)$ and $n_w(t)$ in equations 129 and 130, can be approximated as "white noise". This approximation results in noise autocorrelation functions of the form

$$\overline{n(t)n(t+\tau)} = G_o \delta(\tau) \ , \qquad (160)$$

where $\delta(\tau)$ is the Dirac delta function. This approximation at first may seem unrealistic in that "true" white noise, implying infinite average power, can obviously never exist in nature. The tracker noise, however, can logically be expected to have a very short correlation time relative to the observation time and variations of other functions in equations 148 and 149. Thus the tracker noise autocorrelation function, to a very good approximation, will behave as a "sifting" function, as the Dirac delta function does, in these equations. Further assumptions are that $n_u(t)$ and $n_w(t)$ are statistically independent of each other and of all terms in Ψ_u and Ψ_w, and that gyro drift rates are statistically independent of each other. Tracker noise functions and gyro drift rates do not represent true individual gyro drift rates or actual tracker channel errors; they are derived from these, however, by orthogonal transformations (successive rotations). Then, assuming that all true drift rates exhibit the same statistical properties (and that actual tracker channel errors have the same statistical properties) and all are independent, it follows that the orthogonal transformations preserve independence. For example, the epq drift rates can be written in terms of the xyz, or actual gyro drift rates, in vector form as

$$\underline{\varepsilon}_e(t) = \Theta \, \underline{\varepsilon}_p(t) \ , \qquad (161)$$

where $\underline{\varepsilon}_e$ and $\underline{\varepsilon}_p$ are column vectors whose elements are ε_e, ε_p, ε_q and ε_x

ε_y, ε_z respectively, and Θ is the orthogonal transformation matrix. The general covariance matrix for ε_e can then be written

$$C_e = \overline{\underline{\varepsilon}_e(t)[\underline{\varepsilon}_e(t+\tau)]^T} , \tag{162}$$

where the superscript T denotes transpose. From equation 161, equation 162 can be written

$$C_e = \overline{[\ \Theta\ \underline{\varepsilon}_p(t)][\ \Theta\ \underline{\varepsilon}_p(t+\tau)]^T}$$

$$= \Theta\ \overline{[\underline{\varepsilon}_p(t)][\underline{\varepsilon}_p(t+\tau)]^T}\ \Theta^T\ . \tag{163}$$

Now, since the elements of $\underline{\varepsilon}_p$ are assumed to be independent of each other and to possess the same statistical properties, the covariance matrix for $\underline{\varepsilon}_p$ can be written

$$\overline{[\underline{\varepsilon}_p(t)][\underline{\varepsilon}_p(t+\tau)]^T} = I\ (\overline{\varepsilon(t)\varepsilon(t+\tau)}) \quad , \tag{164}$$

where I is the identity or unit matrix and $\overline{\varepsilon(t)\varepsilon(t+\tau)}$ is a scalar quantity. Substitution of equation 164 into equation 163 gives

$$C_e = \Theta\ I\ \ \Theta^T(\overline{\varepsilon(t)\varepsilon(t+\tau)})$$

$$= I\ (\overline{\varepsilon(t)\varepsilon(t+\tau)}) \quad , \tag{165}$$

since the scalar $(\overline{\varepsilon(t)\varepsilon(t+\tau)})$ can be factored out and $\Theta\ \Theta^T = I$. Thus ε_e, ε_p and ε_q are statistically independent and possess the same autocorrelation functions because all off-diagonal terms of $\underline{C}_e$ are zero and diagonal terms are identical. It is obvious that this same proof applies to $n_u(t)$ and $n_w(t)$.

The general solution for equations 148 and 149, under the approximations and assumptions given, will now be derived. First, it is noted that, under the approximation of constant gyro drift rates, $\Psi_u(t)$ and $\Psi_w(t)$ are given by equations 80 and 81 for the normal operating mode, and by equations 127 and 128 for the decoupled mode. For convenience, both sets of equations will be written

$$\Psi_u(t) = \sum_{j=1}^{n} s_j \, f_{uj}(t) \ , \tag{166}$$

$$\Psi_w(t) = \sum_{j=1}^{n} s_j \, f_{wj}(t) \ . \tag{167}$$

In the normal operating mode, n=5, s_j represents $\Psi_{q'o}$, $\Psi_{po}\cos\eta - \Psi_{e'o}\sin\eta$, $\varepsilon_{q'}$, ε_p, and $\varepsilon_{e'}$ as j ranges from one to five, and the time functions f_{uj} and f_{wj} represent the appropriate time functions in equations 80 and 81. In the decoupled mode, n=2, $s_1 = \Psi_{od}$, $s_2 = \Delta V'$ and the time functions f_{ui} and f_{wi} again represent the appropriate functions in equations 127 and 128. Now, equations 129 and 130 are used to give Ψ_{ub} and Ψ_{wb} from equations 166 and 167, and from these equations the covariance functions of equations 148 and 149 can be written in the manner shown:

$$\overline{\Psi_{ub}(\tau)\Psi_{ub}(x)} = \overline{\left(\sum_{j=1}^{n} s_j f_{uj}(\tau) + n_u(\tau)\right)\left(\sum_{j=1}^{n} s_j f_{uj}(x) + n_u(x)\right)}$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{n} \overline{s_j s_k} \, f_{uj}(\tau) f_{uk}(x) + G_o\delta(\tau - x) \quad , \tag{168}$$

where the distributive property of the averaging process and independence of signals and noise are utilized. Other covariance functions are

$$\overline{\Psi_{ub}(\tau)\Psi_{wb}(x)} = \sum_{j=1}^{n}\sum_{K=1}^{n} \overline{s_j s_k} \, f_{uj}(\tau) f_{wk}(x) \quad , \tag{169}$$

$$\overline{s_i(t)\Psi_{ub}(\tau)} = \sum_{j=1}^{n} \overline{s_i s_j} \, f_{uj}(\tau) \quad , \tag{170}$$

$$\overline{\Psi_{wb}(\tau)\Psi_{wb}(x)} = \sum_{j=1}^{n} \sum_{k=1}^{n} \overline{s_j s_k}\, f_{wj}(\tau) f_{wK}(x) + G_o \delta(\tau - x) \quad , \tag{171}$$

$$\overline{\Psi_{ub}(x)\Psi_{wb}(\tau)} = \sum_{j=1}^{n} \sum_{k=1}^{n} \overline{s_j s_K}\, f_{uj}(x) f_{wK}(\tau) \quad , \tag{172}$$

and

$$\overline{s_i(t)\Psi_{wb}(\tau)} = \sum_{j=1}^{n} \overline{s_i s_j}\, f_{wj}(\tau) \quad . \tag{173}$$

Equations 168 through 173 are now substituted into equations 148 and 149, giving

$$\sum_{j=1}^{n} \overline{s_i s_j}\, f_{uj}(\tau) = \int_o^t [w_{i1}(t,x)\Big(\sum_{j=1}^{n} \sum_{K=1}^{n} \overline{s_j s_K}\, f_{uj}(\tau) f_{uK}(x) + G_o \delta(\tau - x) \Big)]dx$$
$$+ \int_o^t [w_{i2}(t,x)\Big(\sum_{j=1}^{n} \sum_{K=1}^{n} \overline{s_j s_K}\, f_{uj}(\tau) f_{wK}(x) \Big)]dx \ , \ 0 \leq \tau \leq t, \tag{174}$$

and

$$\sum_{j=1}^{n} \overline{s_i s_j}\, f_{wj}(\tau) = \int_o^t [w_{i2}(t,x)\Big(\sum_{j=1}^{n} \sum_{K=1}^{n} \overline{s_j s_K}\, f_{wj}(\tau) f_{wK}(x) + G_o \delta(\tau - x) \Big)]dx$$
$$+ \int_o^t w_{i1}(t,x)\Big(\sum_{j=1}^{n} \sum_{K=1}^{n} \overline{s_j s_K}\, f_{wj}(\tau) f_{uK}(x) \Big) dx \ , \ 0 \leq \tau \leq t, \tag{175}$$

where the "o" subscript has been (and will be in the future) dropped from the weighting functions. These equations, although formidable in appearance, have a relatively simple and quite straightforward solution. First, the "sifting" property of the Dirac delta function is invoked. This property gives the result

$$\int_0^t w_{i1}(t,x)\ (Go\delta(\tau-x))dx = Go\ w_{i1}(t,\tau)\ ,\ 0 \leq \tau \leq t\ , \tag{176}$$

and an identical result for w_{i2}. Then equations 174 and 175 can be written

$$\sum_j \overline{s_i s_j}\ f_{uj}(\tau) - \sum_j \left\{ f_{uj}(\tau) \int_0^t w_{i1}(t,x)\left(\sum_k \overline{s_j s_K}\ f_{uK}(x)\right) dx \right\}$$

$$-\sum_j \left\{ f_{uj}(\tau) \int_0^t w_{i2}(t,x)\left(\sum_k \overline{s_j s_K}\ f_{wK}(x)\right) dx \right\}$$

$$= Go\ w_{i1}(t,\tau)\ ,\ 0 \leq \tau \leq t\ , \tag{177}$$

and

$$\sum_j \overline{s_i s_j}\ f_{wj}(\tau) - \sum_j \left\{ f_{wj}(\tau) \int_0^t w_{i2}(t,x)\left(\sum_k \overline{s_j s_K}\ f_{wK}(x)\right) dx \right\}$$

$$-\sum_j \left\{ f_{wj}(\tau) \int_0^t w_{i1}(t,x)\left(\sum_K \overline{s_j s_K}\ f_{uK}(x)\right) dx \right\}$$

$$= Go\ w_{i2}(t,\tau)\ ,\ 0 \leq \tau \leq t\ . \tag{178}$$

Inspection of equations 177 and 178 shows that they can be written

$$w_{i1}(t,\tau) = \sum_{j=1}^{n} K_{iuj}(t)\ f_{uj}(\tau)\ , \tag{179}$$

$$w_{i2}(t,\tau) = \sum_{j=1}^{n} K_{iwj}(t)\ f_{wj}(\tau)\ , \tag{180}$$

because the only functions of τ appearing on the left-hand sides of these equations are f_{uj} and f_{wj}, respectively. The only unknowns remaining, then, are the functions $K_{iuj}(t)$ and $K_{iwj}(t)$. These are found by substituting equations 179 and 180 into 177 and 178 and carrying out the indicated integrations. This can be done because K_{iuj} and K_{iwj}, being functions of t, will factor out of the integrals. The results of these operations can be written

$$Go \sum_{j=1}^{n} K_{iuj}(t)\, f_{uj}(\tau) - \sum_{j=1}^{n} f_{uj}(\tau) \Bigg[\overline{s_i s_j}$$

$$- \sum_{k=1}^{n} \left(K_{iuk}(t)\, h_{uk}(t) + K_{iwK}(t)\, h_{wk}(t) \right) \Bigg] = 0 \; ,$$

$$0 \leq \tau \leq t \qquad (181)$$

and

$$Go \sum_{j=1}^{n} K_{iwj}(t)\, f_{wj}(\tau) - \sum_{j=1}^{n} f_{wj}(\tau) \Bigg[\overline{s_i s_j}$$

$$- \sum_{k=1}^{n} \left(K_{iwk}(t)\, h_{wk}(t) + K_{iuk}(t)\, h_{uk}(t) \right) \Bigg] = 0 \; ,$$

$$0 \leq \tau \leq t \; . \qquad (182)$$

Now, equations 181 and 182 must be satisfied for all τ, $0 \leq \tau \leq t$, and therefore are identities in τ. The coefficients of each of the functions f_{uj} in equation 181 and f_{wj} in 182 must therefore be zero if the functions $\{f_{uj}\}$ (and $\{f_{wj}\}$) are linearly independent, which they are in this case. Setting these coefficients equal to zero gives a set of 2n equations in the 2n unknowns, K_{iuj} and K_{iwj}, $j=1,2,\cdots,n$; thus these terms can be found and the optimum filters are then completely specified. It should be noted that the covariances $\overline{s_i s_j}$ are assumed known. These are ordinarily found by laboratory tests on gyros and the sun-tracker, and computations to determine quantities involving components of $\underline{\Psi}_o$. Also, the functions $h_{uk}(t)$ and $h_{wk}(t)$ represent values of the integrals in equations 177 and 178.

A simple example will serve to clarify this method of solution. Consider the solution for the optimum filter for Ψ_{po} when $\eta=0$. Equation 81 then reduces to

$$\Psi_w(t) = \Psi_{po} + \varepsilon_p t \quad . \tag{183}$$

It is also observed that $\overline{\Psi_{wb}(\tau)\Psi_{ub}(x)} \equiv 0$ and therefore $\overline{\Psi_{po}\Psi_{ub}(\tau)} \equiv 0$. Equation 148 can therefore be ignored, as might be suspected, since Ψ_{ub} contains no information as to Ψ_{wb}. Equation 149 reduces to

$$\int_0^t w_2(t,x)\ \overline{\Psi_{wb}(\tau)\Psi_{wb}(x)}\ dx = \overline{\Psi_{po}\Psi_{wb}(\tau)}\ ,\ 0 \le \tau \le t\ ,$$

or

$$\int_0^t w_2(t,x)\ [\overline{\Psi_{po}^{\ 2}} + \overline{\varepsilon_p^{\ 2}}\ \tau x + \overline{\Psi_{po}\varepsilon_p}(\tau+x)]dx$$

$$+ \text{Go}\ w_2(t,\tau) = \overline{\Psi_{po}^{\ 2}} + \overline{\Psi_{po}\varepsilon_p}\ \tau\ ,\ 0 \le \tau \le t\ , \tag{184}$$

where the covariances have been expanded. Equation 184 can be rewritten in the form

$$\text{Go}\ w_2(t,\tau) = \left[\overline{\Psi_{po}^{\ 2}} - \int_0^t w_2(t,x)(\overline{\Psi_{po}^{\ 2}} + \overline{\Psi_{po}\varepsilon_p}x)dx\right]$$

$$+ \tau\left[\overline{\Psi_{po}\varepsilon_p} - \int_0^t w_2(t,x)(\overline{\varepsilon_p^{\ 2}}x + \overline{\Psi_{po}\varepsilon_p})dx\right],\quad 0 \le \tau \le t \tag{185}$$

Since the terms in the large brackets are functions of t only, it is clear that $w_2(t,\tau)$ has the form

$$w_2(t,\tau) = K_1(t) + \tau\ K_2(t) \quad . \tag{186}$$

Equation 186 is now substituted into equation 185 and the indicated integrations carried out, giving

$$\begin{aligned}
Go(K_1(t) + \tau K_2(t)) &= \Bigg[\overline{\Psi_{po}^2} - K_1(t)\left(\overline{\Psi_{po}^2}\, t + \overline{\Psi_{po}\varepsilon_p}\, \frac{t^2}{2}\right) \\
&- K_2(t)\left(\overline{\Psi_{po}^2}\, \frac{t^2}{2} + \overline{\Psi_{po}\varepsilon_p}\, \frac{t^3}{3}\right)\Bigg] \\
&+ \tau \Bigg[\overline{\Psi_{po}\varepsilon_p} - K_1(t)\left(\overline{\Psi_{po}\varepsilon_p}\, t + \overline{\varepsilon_p^2}\, \frac{t^2}{2}\right) \\
&- K_2(t)\left(\overline{\Psi_{po}\varepsilon_p}\, \frac{t^2}{2} + \overline{\varepsilon_p^2}\, \frac{t^3}{3}\right)\Bigg], \quad 0 \le \tau \le t \quad . \qquad (187)
\end{aligned}$$

It is now clear that the terms independent of τ must be equal, and the terms linear in τ must also be equal. This gives two equations in the two unknowns, K_1 and K_2, which can be written in matrix form

$$\begin{pmatrix} \left(Go + \overline{\Psi_{po}^2}\, t + \overline{\Psi_{po}\varepsilon_p}\, \frac{t^2}{2}\right) & \left(\overline{\Psi_{po}^2}\, \frac{t^2}{2} + \overline{\Psi_{po}\varepsilon_p}\, \frac{t^3}{3}\right) \\ \left(\overline{\Psi_{po}\varepsilon_p}\, t + \overline{\varepsilon_p^2}\, \frac{t^2}{2}\right) & \left(Go + \overline{\Psi_{po}\varepsilon_p}\, t + \overline{\varepsilon_p^2}\, \frac{t^2}{2}\right) \end{pmatrix} \begin{pmatrix} K_1(t) \\ K_2(t) \end{pmatrix} = \begin{pmatrix} \overline{\Psi_{po}^2} \\ \overline{\Psi_{po}\varepsilon_p} \end{pmatrix} \qquad (188)$$

The solution for K_1 and K_2 is then quite straightforward.

Other examples are given in Section VII. As can be seen, the method can become ungainly; for the normal operating mode, five signals are present in general (n=5) and it is then necessary to invert a 10x10 matrix. The determinant of the matrix could have as many as 10 ! terms (although some simplification is certainly expected). At any rate, such weighting functions would certainly tax the capacity of a typical computer; further simplifications are therefore made in Section VII.

Before proceeding to this, however, one more general problem remains. In the work presented thus far, it has been assumed that no corrections will be applied to the system during sun observations. Since the filters derived give the minimum mean-square error in estimating system errors at any time after observations commence, it is logically quite desirable to make system corrections continuously, rather than waiting a fixed length of time. Such a procedure, however, would seem to invalidate the solutions for the optimum filters which have been derived, since $\underline{\Psi}$ and drift rates would then change as corrections are made. A mechanization method which allows continuous corrections to be made while still preserving the validity of the filter derivation is derived in the next Section.

VI. CONTINUOUS ERROR CORRECTION MECHANIZATION

The continuous error-correction mechanization derived here relies on the fact that superposition is valid within the linear system under consideration. Before the derivation is presented, it is helpful to represent the system, as mechanized so far, by the block diagram of Figure 8. This diagram represents the system in matrix notation for the normal operating mode. Thus all quantities "flowing" from block to block are column vectors, and the blocks in general represent matrices which operate on these vectors. The block showing $\int_0^t (\ )$ indicates that each component of the vector input is integrated. The matrix $\underline{\underline{\Omega}}$ represents the "feed-back" present in equations 45, and in full form (for the e'pq' system is

$$\underline{\underline{\Omega}} = \begin{pmatrix} 0 & 0 & -\Omega \\ 0 & 0 & 0 \\ \Omega & 0 & 0 \end{pmatrix} \quad . \tag{189}$$

It can then be seen that this section of the diagram represents equations 45 in block diagram, matrix form. This form, incidentally, is the standard form used by Kalman and Bucy (10) in their approach to the problem. The matrix $\underline{\underline{B}}(t)$ represents a time-varying gain and is the matrix representation of equations 78 and 79. Thus

$$\underline{\Psi}_b(t) = \underline{n}(t) + \underline{\underline{B}}(t)\underline{\Psi}(t) \quad , \tag{190}$$

where

$$\underline{\Psi}_b(t) = \begin{pmatrix} \Psi_{ub}(t) \\ \Psi_{wb}(t) \end{pmatrix} \quad , \tag{191}$$

$$\underline{n}(t) = \begin{pmatrix} n_u(t) \\ n_w(t) \end{pmatrix} \tag{192}$$

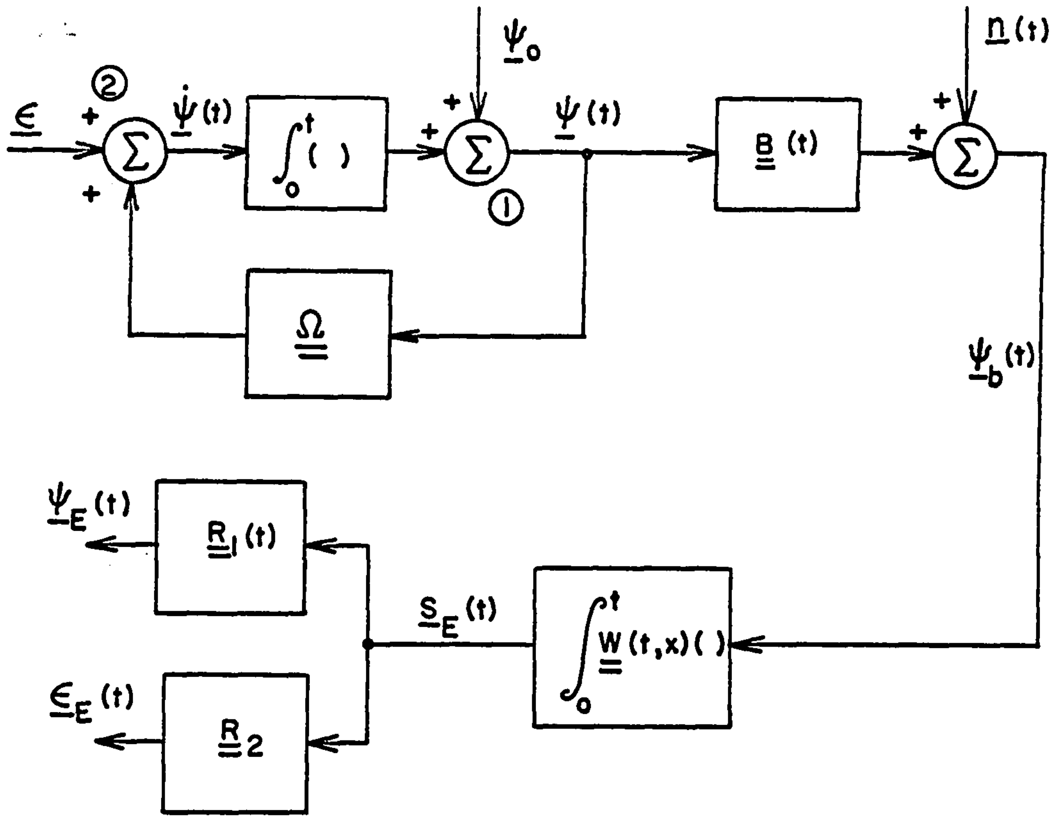

Figure 8. Block diagram of error propagation and filter system-normal operating mode

$$\underline{\Psi}(t) = \begin{pmatrix} \Psi_{e'}(t) \\ \Psi_p(t) \\ \Psi_{q'}(t) \end{pmatrix} , \qquad (193)$$

and

$$\underline{\underline{B}}(t) = \begin{pmatrix} (-\sin\Omega t) & 0 & (\cos\Omega t) \\ (-\sin\eta\cos\Omega t) & (\cos\eta) & (-\sin\eta\sin\Omega t) \end{pmatrix} . \qquad (194)$$

The block containing the term $\int_o^t \underline{\underline{w}}(t,x)(\ )$ represents the optimum filters, where $\underline{\underline{w}}(t,x)$ is a 2x5 matrix, given by

$$\underline{\underline{w}}(t,x) = \begin{pmatrix} w_{11}(t,x) & w_{12}(t,x) \\ w_{21}(t,x) & w_{22}(t,x) \\ \vdots & \vdots \\ w_{51}(t,x) & w_{52}(t,x) \end{pmatrix} , \qquad (195)$$

and the output, $\underline{s}_E(t)$, is the estimated signal vector:

$$\underline{s}_E(t) = \begin{pmatrix} \underline{s}_1(t) \\ \underline{s}_2(t) \\ \vdots \\ s_5(t) \end{pmatrix} = \begin{pmatrix} \Psi_{q'oE} \\ \varepsilon_{q'E} \\ \varepsilon_{e'E} \\ \Psi_{poE}\cos\eta - \Psi_{e'oE}\sin\eta \\ \varepsilon_{pE} \end{pmatrix} , \qquad (196)$$

This block is then meant to represent the operation

$$\underline{s}_E(t) = \int_o^t \underline{\underline{w}}(t,x)\underline{\Psi}_b(x)dx \ . \qquad (197)$$

The blocks $\underline{\underline{R}}_1(t)$ and $\underline{\underline{R}}_2$ represent the mathematical operations necessary to reconstruct estimates of $\underline{\Psi}(t)$ and $\underline{\varepsilon}_e(t)$ respectively. Thus

$$\underline{\underline{R}}_1(t) = \begin{pmatrix} (-\sin\Omega t) & (-\frac{1}{\Omega}(1-\cos\Omega t)) & (\frac{\sin\Omega t}{\Omega}) & (-\sin\eta\cos\Omega t) & 0 \\ 0 & 0 & 0 & \cos\eta & t \\ (\cos\Omega t) & (\frac{\sin\Omega t}{\Omega}) & (\frac{1}{\Omega}(1-\cos\Omega t) & (-\sin\eta\sin\Omega t) & 0 \end{pmatrix}, \qquad (198)$$

and

$$\underline{\underline{R}}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \qquad (199)$$

where $\underline{\underline{R}}_1(t)$ is derived from equations 73, 74 and 75 and $\underline{\underline{R}}_2$ is obvious by inspection of equation 196. Thus best (in a least-square sense) estimates of $\underline{\Psi}(t)$ and $\underline{\varepsilon}$ are available at all times.

The problem now is what to do with these estimates. Since the basic purpose of this study is to minimize the errors within the system, an obvious step is to apply $\underline{\Psi}_e(t)$ and $\underline{\varepsilon}_e(t)$ in a negative sense to summing points 1 and 2 respectively. This step alone, however, is not valid as $\underline{\Psi}(t)$ then would no longer have the form assumed for it in finding $\underline{w}(t,x)$. This difficulty can be overcome by recognizing that $\underline{\Psi}_e(t)$ and $\underline{\varepsilon}_e(t)$ are known at all times within the computer. Thus, corrections can be applied continuously to the actual system if the effects of these corrections are removed from $\underline{\Psi}_b(t)$ before it is processed by the optimum filter. This can be done within the system computer by constructing a model of the dynamical system, applying $\underline{\Psi}_e(t)$ and $\underline{\varepsilon}_e(t)$ to this model, and removing by subtraction the component of $\underline{\Psi}_b(t)$ generated by these corrections from the actual $\underline{\Psi}_b(t)$. This mechanization is shown in Figure 9, where all elements within the dashed lines represent computer operations. The actual observed quantity is now denoted as $\underline{\Psi}_{b'}(t)$. It can be seen

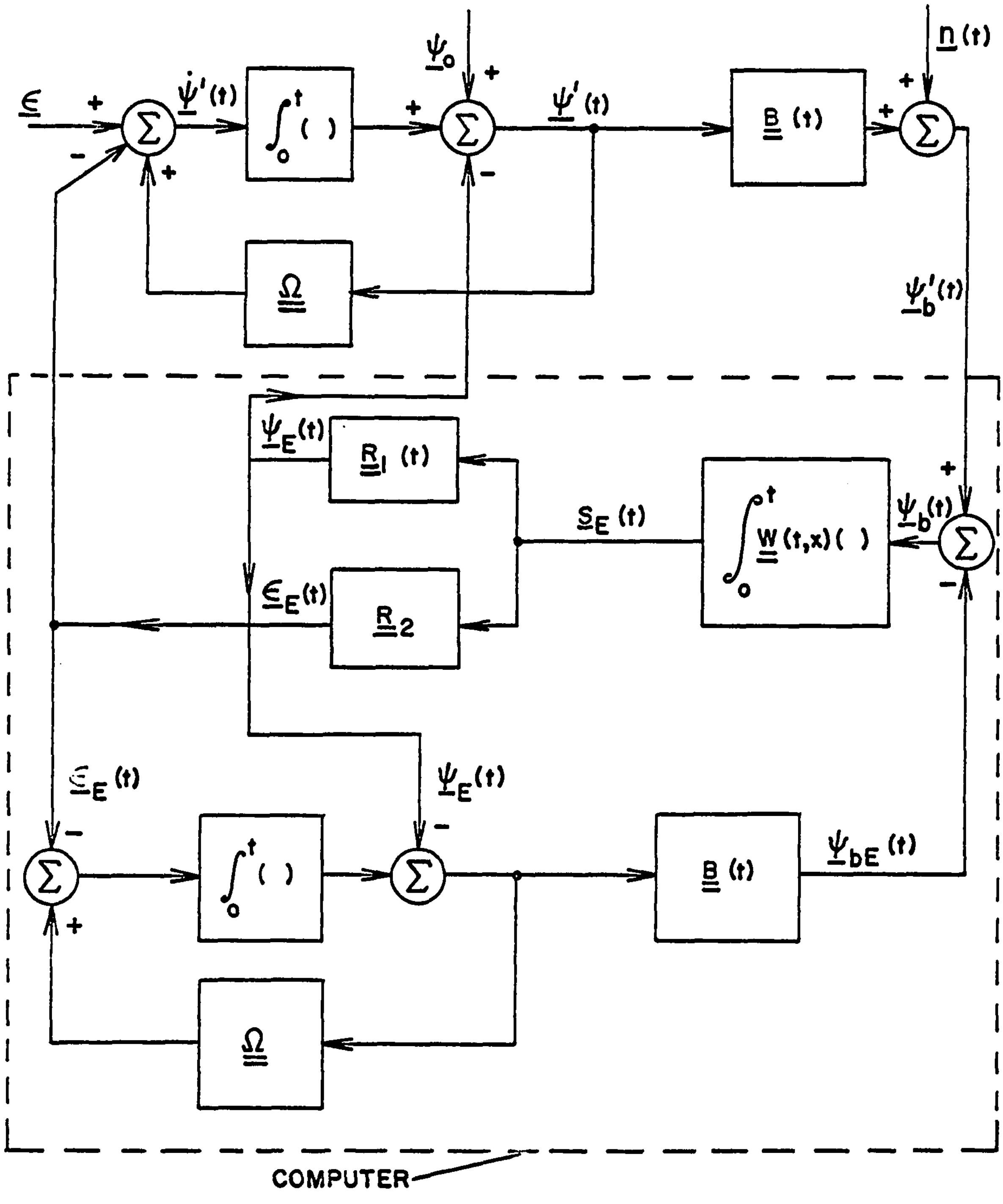

Figure 9. System block diagram showing continuous error correction mechanization

from Figure 9 that identical operations are carried out on $\underline{\Psi}_E$ and $\underline{\varepsilon}_E$ in both the actual dynamical system and the model within the computer. Since the system is linear and superposition applies, $\underline{\Psi}_{b'}(t)$ can be written

$$\underline{\Psi}_{b'}(t) = \underline{\Psi}_b(t) + \underline{\Psi}_{bE}(t) \quad , \tag{200}$$

where $\underline{\Psi}_b(t)$ is the "old" observed signal which would be present with no corrections applied, and $\underline{\Psi}_{bE}(t)$ is the component of $\underline{\Psi}_{b'}$ due to the corrections. Thus $\underline{\Psi}_{bE}(t)$ is the signal which would be present with $\underline{\varepsilon}$, $\underline{\Psi}_o$ and $\underline{n}$ all zero. This is obviously the signal generated by the model within the computer; subtraction of this signal from $\underline{\Psi}_{b'}(t)$ (as indicated in Figure 9) therefore yields $\underline{\Psi}_b(t)$ as the input to the filters. The quantity $\underline{\Psi}\ (t)$ in Figure 9, which is the actual Ψ angle, is thus minimized (in a least square sense) for all times after observations commence.

This technique, then, is a method of designing an optimum closed-loop or regulator system containing linear time-varying elements from a corresponding optimum open-loop or error estimation system. All elements within the dashed lines could in principal be combined into one time-varying linear filter. This is extremely difficult to do in practice, however, because the system contains time-varying elements in feed-back loops. Thus conventional feed-back theory is not applicable, and the solution for such a filter is equivalent to the solution of a time-varying, linear, matrix differential equation. Since a computer implementation of the model as shown in Figure 9 is a simple task, it will be left as is.

As was pointed out earlier, the general equations for the optimum filter derived in Section V, although relatively simple in form, very likely cannot be accommodated within a typical computer. Further simplifications are therefore necessary; these are discussed in the next Section and the resulting filter weighting functions are explicitly derived. Since system errors are of prime importance, an error analysis is also made.

VII. EXPLICIT SOLUTIONS FOR OPTIMUM FILTERS AND ERROR ANALYSIS

The first set of optimum filters to be derived are those for the normal operating mode; that is, those giving estimates of $\Psi_{q'o}$, $\Psi_{po} \cos\eta - \Psi_{e'o} \sin\eta$, $\varepsilon_{e'}$, ε_p and $\varepsilon_{q'}$. It has been indicated previously that equations 80 and 81 will be used as models for Ψ_u and Ψ_w. Inspection of these equations shows that only terms common to them are $\varepsilon_{e'}$ and $\varepsilon_{q'}$, and that in equation 81 $\varepsilon_{e'}$ and $\varepsilon_{q'}$ have as multipliers $\sin\eta$. The maximum value of η is about 23.5 degrees. Thus the maximum value of $\sin\eta$ is about 0.4, and its mean-square value (averaged over a year) is about 0.08. This indicates that $\Psi_w(t)$ will not be very useful in estimating either $\varepsilon_{e'}$ or $\varepsilon_{q'}$. Since no other terms are common to Ψ_u and Ψ_w, it was decided to use Ψ_{ub} only for estimating $\Psi_{q'o}$, $\varepsilon_{q'}$ and $\varepsilon_{e'}$, and Ψ_{wb} only to estimate the remaining two terms. This admittedly gives rise to sub-optimum estimates, but results in much simpler forms for the filters. It is also expected that, because of the weak coupling discussed above, these estimates will be quite close to the true optimum.

It was also stated that gyro drift rates are approximated as constants because the observation time is expected to be of the order of one or two hours-short compared to drift rate correlation time. It seems quite reasonable then to approximate $\sin\Omega t$ by Ωt and $\cos\Omega t$ by $1 - \frac{\Omega^2 t^2}{2}$ in equations 80 and 81, in order to simplify the forms of the optimum filters. The exact expressions could be used in the filters without increasing their complexity too much; the corresponding error equations, however, become more difficult to handle and the approximations are certainly valid in computing errors.

With these approximations, then, equations 80 and 81 give the following equations:

$$\Psi_{ub}(t) = \Psi_{q'o} + \varepsilon_{q'}t - \frac{\Omega}{2}\varepsilon_{e'}t^2 + n_u(t) \tag{201}$$

$$\Psi_{wb}(t) = (\Psi_{po}\cos\eta - \Psi_{e'o}\sin\eta) + (\varepsilon_p\cos\eta - \varepsilon_{e'}\sin\eta)t$$
$$- \frac{\Omega}{2}\varepsilon_{q'}\sin\eta\, t^2 + n_w(t) \quad . \tag{202}$$

Two things should be noted at this point. First, in equation 202, ε_p and $\varepsilon_{e'}$ are not separable; $\varepsilon_{e'}$, however, is available in equation 201, and ε_p can therefore be found explicitly if desired. Second, both equations are of the form shown below:

$$\Psi_b(t) = s_1 + s_2 t + \frac{\Omega}{2} s_3 t^2 + n(t) \tag{203}$$

Thus, solution for the optimum filters for equation 203 will give, with proper substitutions, the filters for both equations 201 and 202. These optimum filters will be found first for the case of no correlation between random variables. This case is the simpler, and represents an actual situation which could arise, for example, if a system were initially put into operation at sea and coarse initial alignment achieved.

The solution proceeds in the manner outlined in Section VI. Only one integral equation is significant here. This can be written

$$\int_0^t w_i(t,x)\, \overline{\Psi_b(\tau)\Psi_b(x)}\, dx = \overline{s_i\Psi_b(\tau)} \; , \; 0 \leq \tau \leq t \tag{204}$$

The function $\overline{\Psi_b(\tau)\Psi_b(x)}$, for no correlation, has the simple form

$$\overline{\Psi_b(\tau)\Psi_b(x)} = \overline{s_1^{\,2}} + \overline{s_2^{\,2}}\,\tau x + \frac{\Omega^2}{4}\,\overline{s_3^{\,2}}\,\tau^2 x^2 + G_o\delta(\tau - x) \tag{205}$$

$$= A + B\,\tau x + C\,\tau^2 x^2 + G_o \delta(\tau - x) \quad , \tag{206}$$

where the symbols A, B and C have obvious meanings. The function $\overline{s_i \Psi_b(\tau)}$ varies with s_i; the three forms are

$$\overline{s_1 \Psi_b(\tau)} = \overline{s_1^{\,2}} = A \tag{207}$$

$$\overline{s_2 \Psi_b(\tau)} = \overline{s_2^{\,2}}\ \tau = B\ \tau \tag{208}$$

$$\overline{s_3 \Psi_b(\tau)} = \frac{\Omega}{2}\ \overline{s_3^{\,2}}\ \tau^2 = \frac{2}{\Omega}\ C\ \tau^2 \quad . \tag{209}$$

Substitution of equation 205 into 204 gives

$$A \int_o^t w_i(t,x)dx + B\ \tau \int_o^t x\ w_i(t,x)dx$$

$$+ C\ \tau^2 \int_o^t x^2\ w_i(t,x)dx + G_o w_i(t,\tau)$$

$$= \overline{s_i \Psi_b(\tau)}\ ,\ 0 \le \tau \le t \quad . \tag{210}$$

Inspection of equations 207, 208 and 209 shows that, for i=1,2 or 3, $w_i(t,\tau)$ has the form

$$w_i(t,\tau) = K_{i1}(t) + \tau K_{i2}(t) + \tau^2 K_{i3}(t) \quad . \tag{211}$$

When the indicated integrations are performed, equation 210 takes the form

$$A(K_{i1} t + K_{i2}\frac{t^2}{2} + K_{i3}\frac{t^3}{3}) + B\ \tau\ (K_{i1}\frac{t^2}{2} + K_{i2}\frac{t^3}{3} + K_{i3}\frac{t^4}{4})$$

$$+ C\ \tau^2 (K_{i1}\frac{t^3}{3} + K_{i2}\frac{t^4}{4} + K_{i3}\frac{t^5}{5}) + G_o(K_{i1} + K_{i2}\ \tau + K_{i3}\ \tau^2)$$

$$= \overline{s_i \Psi_b(\tau)}\ ,\quad 0 \le \tau \le t \quad , \tag{212}$$

and, by the reasoning of Section V, equation 212 yields three equations in K_{i1}, K_{i2} and K_{i3}, which can be written in matrix form:

$$\begin{pmatrix} (G_o + At) & (A\frac{t^2}{2}) & (A\frac{t^3}{3}) \\ (B\frac{t^2}{2}) & (G_o + B\frac{t^3}{3}) & (B\frac{t^4}{4}) \\ (C\frac{t^3}{3}) & (C\frac{t^4}{4}) & (G_o + C\frac{t^5}{5}) \end{pmatrix} \begin{pmatrix} K_{i1} \\ K_{i2} \\ K_{i3} \end{pmatrix} = \underline{V}_i \quad , \tag{213}$$

where $\underline{V}_i$ is a column vector representing the coefficients of $\overline{s_i \Psi_b}(\tau)$:

$$\underline{V}_1 = \begin{pmatrix} A \\ 0 \\ 0 \end{pmatrix} \tag{214}$$

$$V_2 = \begin{pmatrix} 0 \\ B \\ 0 \end{pmatrix} \tag{215}$$

$$V_3 = \begin{pmatrix} 0 \\ 0 \\ \frac{2C}{\Omega} \end{pmatrix} \tag{216}$$

It is now a simple matter to solve for the K's. Since the matrix is a 3x3 Cramer's rule is used. It is noted that the determinant of the 3x3 matrix in equation 213 (hereafter denoted as $\underline{\underline{T}}$) will be the denominator for all nine K's. This determinant, called Δ, has the value

$$\Delta = G_o^{\,3} + G_o^{\,2} At + G_o^{\,2} B \frac{t^3}{3} + G_o A B \frac{t^4}{12} + G_o^{\,2} C \frac{t^5}{5} + G_o A C (\frac{4t^6}{45}) + G_o B C \frac{t^8}{240} + A B C \frac{t^9}{2160} \quad . \tag{217}$$

Solutions for the K's are:

$$K_{11} = \frac{A}{\Delta}[G_o{}^2 + G_o\ B\ \frac{t^3}{3} + G_o\ C\ \frac{t^5}{5} + \frac{BCt^8}{240}] \tag{218}$$

$$K_{12} = -\frac{B}{\Delta}[G_o\ A\ \frac{t^2}{2} + A\ C\ \frac{t^7}{60}] \tag{219}$$

$$K_{13} = \frac{C}{\Delta}[-G_o\ A\ \frac{t^3}{3} + A\ B\ \frac{t^6}{72}] \tag{220}$$

$$K_{21} = -\frac{A}{\Delta}[G_o\ B\ \frac{t^2}{2} + A\ C\ \frac{t^7}{60}] \tag{221}$$

$$K_{22} = \frac{B}{\Delta}[G_o{}^2 + G_o\ At + G_o\ C\frac{t^5}{5} + A\ C\ (\frac{4t^6}{45})] \tag{222}$$

$$K_{23} = -\frac{C}{\Delta}[G_o\ B\ \frac{t^4}{4} + A\ B\ \frac{t^5}{12}] \tag{223}$$

$$K_{31} = \frac{2A}{\Omega\Delta}[-G_o\ C\ \frac{t^3}{3} + B\ C\ \frac{t^6}{72}] \tag{224}$$

$$K_{32} = -\frac{2B}{\Omega\Delta}[G_o\ C\ \frac{t^4}{4} + A\ C\ \frac{t^5}{12}] \tag{225}$$

$$K_{33} = \frac{2C}{\Omega\Delta}[G_o{}^2 + G_o\ At + G_o\ B\ \frac{t^3}{3} + A\ B\ \frac{t^4}{12}] \quad . \tag{226}$$

The optimum filters are thus specified explicitly for this case. It is interesting to note that the mechanization of these filters within a computer is quite simple. Since the K's are independent of the integration process, they can be regarded as time-varying gains. Thus only six quantities need by integrated; $\Psi_{ub}(x)$, $x\Psi_{ub}(x)$, $x^2\Psi_{ub}(x)$, $\Psi_{wb}(x)$, $x\Psi_{wb}(x)$ and $x^2\Psi_{wb}(x)$. These quantities are then multiplied by the appropriate K's and combined to form the optimum estimates. For example, the estimate of $\Psi_{q'o}$ is formed as

$$\Psi_{q'oE} = K_{1u1}(t)\int_o^t \Psi_{ub}(x)dx + K_{1u2}(t)\int_o^t x\Psi_{ub}(x)dx + K_{1u3}(t)\int_o^t x^2\Psi_{ub}(x)dx \quad . \tag{227}$$

Of paramount importance in the synthesis of a system such as this is system error. The optimum filters which have been derived do not, of course, drive errors to zero because of the random tracker noise present; the estimation errors must therefore be computed. Equation 142 gives the general form for mean-square estimation error. In the simplified case considered here, w_{i2} can be set equal to zero, giving

$$\overline{e_{si}^2(t)} = \overline{s_i^2} + \int_0^t \int_0^t w_{i1}(t,\tau)\, w_{i1}(t,x)\, \overline{\Psi_{ub}(\tau)\Psi_{ub}(x)}\, d\tau\, dx$$

$$- 2\int_0^t w_{i1}(t,\tau)\, \overline{s_i(t)\Psi_{ub}(\tau)}\, d\tau \ . \tag{228}$$

This equation can be simplified further by recognizing that $w_{i1}(t,x)$ satisfies the integral equation 204. Substitution of equation 204 into 228 yields

$$\overline{e_{si}^2(t)} = \overline{s_i^2} - \int_0^t w_{i1}(t,x)\, \overline{s_i(t)\Psi_{ub}(x)}\, dx \quad , \tag{229}$$

which is the equation to be used in finding errors. Substitution of equations 207, 208, 209 and 211 into 229 gives three equations for errors:

$$\overline{e_{s1}^2(t)} = A[1 - K_{11}t - K_{12}\frac{t^2}{2} - K_{13}\frac{t^3}{3}] \tag{230}$$

$$\overline{e_{s2}^2(t)} = B[1 - K_{21}\frac{t^2}{2} - K_{22}\frac{t^3}{3} - K_{23}\frac{t^4}{4}] \tag{231}$$

$$\overline{e_{23}^2(t)} = \frac{4C}{\Omega^2}[1 - \frac{\Omega}{2}(K_{31}\frac{t^3}{3} + K_{32}\frac{t^4}{4} + K_{33}\frac{t^5}{5})] \ . \tag{232}$$

Substitution of the K's (equations 218 through 226) into these equations gives (after some reduction)

$$\overline{e_{s1}^2(t)} = \frac{G_o A}{\Delta}[G_o^{\,2} + G_o B \frac{t^3}{3} + G_o C \frac{t^5}{5} + B C \frac{t^8}{240}] \tag{233}$$

$$\overline{e_{s2}^2(t)} = \frac{G_o B}{\Delta}[G_o^{\,2} + G_o At + G_o C \frac{t^5}{5} + A C (\frac{4t^6}{45})] \tag{234}$$

$$\overline{e_{s3}^2(t)} = \frac{4G_o C}{\Omega^2}[G_o^{\,2} + G_o At + G_o B \frac{t^3}{3} + A B \frac{t^4}{12}] \quad . \tag{235}$$

It is comforting to note that, for $G_o = 0$ and all $t \neq 0$, all mean-square errors are zero. This indicates, as it should, that it is possible to determine the desired quantities exactly in the absence of noise. The asymptotic behavior of these errors is also of interest, and is found by discarding all but the highest power of t in each numerator and denominator. The results are

$$\overline{e^2_{s1\ \mathrm{ASYM.}}} = \frac{9\ G_o}{t} \tag{236}$$

$$\overline{e^2_{s2\ \mathrm{ASYM.}}} = \frac{192\ G_o}{t^3} \tag{237}$$

$$\overline{e^2_{s3\ \mathrm{ASYM.}}} = \frac{720\ G_o}{\Omega^2 t^5} \quad . \tag{238}$$

These results are quite simple in form and therefore useful for making rough estimates of observation time required to reach a desired mean-square error. It is interesting to note that they are independent of the initial navigation system errors.

At this point, one still does not have a good "feeling" for the practicality of this system. Numerical examples are therefore certainly in order, and are presented in Section VIII. First the error equations for correlated system errors and for the decoupled mode will be derived, as well as errors due to approximations made.

Correlated system errors will be present in any cyclic, or day-to-day system operation. This can be seen by inspection of equations 60. If system operation consists of sun observation from say, 10 A.M. to 2 P.M., followed by conventional damped inertial navigation until the following day, equations 60 give the $\underline{\Psi}$ error propagation forms for the damped inertial mode. Here $\underline{\Psi}_o$ is the residual error in $\underline{\Psi}$ (due to imperfect estimation) existing at 2 P.M., and gyro drift rates can be written

$$\varepsilon(t) = \varepsilon_r(t) - \varepsilon_r(0) + \varepsilon_E \quad , \tag{239}$$

where $\varepsilon_r(t)$ is the (random) gyro drift rate and ε_E is the error in estimating the value of $\varepsilon_r(t)$ at t=0 (commencement of damped inertial navigation). Equations 60, when evaluated for t = 20 hours (10 A.M. the following day) give the initial value of $\underline{\Psi}$ for the next observation period, $\underline{\Psi}_{o\ new}$. It can be seen, then, that the new values of $\Psi_{e'o}$ and $\Psi_{q'o}$ will in general be correlated with $\varepsilon_{e'new}$ and $\varepsilon_{q'new}$, and $\Psi_{po\ new}$ with $\varepsilon_{p\ new}$. Thus, terms of the form $\overline{s_1 s_2}$ and $\overline{s_1 s_3}$ will appear in the expansion of $\overline{\Psi_b(\tau)\Psi_b(x)}$ and $\overline{s_i \Psi_b(\tau)}$. These terms, along with the variance terms $\overline{s_i^{\,2}}$ will be derived from equations 60 and 239. The derivation is quite straightforward but somewhat lengthy, and is therefore presented in Appendix A. It is noted in passing that these equations show the build-up in time of mean-square system errors due to the undamped normal modes of the $\underline{\Psi}$ equation, as was stated earlier.

The solution for the optimum filters, in this case, begins again with equation 204. The covariance functions have the form

$$\overline{\Psi_b(\tau)\Psi_b(x)} = \overline{s_1^2} + \overline{s_2^2}\ \tau x + \overline{s_3^2}\ \tau^2 x^2 + \overline{s_1 s_2}(\tau+x) + \overline{s_1 s_3}(\tau^2+x^2)$$

$$+ G_o \delta(\tau-x) = A + B\ \tau x + C\ \tau^2 x^2 + a(\tau+x) + b(\tau^2+x^2)$$

$$+ G_o \delta(\tau-x) \quad , \tag{240}$$

$$\overline{s_1\Psi_b(\tau)} = \overline{s_1^2} + \overline{s_1 s_2}\ \tau + \overline{s_1 s_3}\ \tau^2 = A + a\tau + b\tau^2 \tag{241}$$

$$\overline{s_2\Psi_b(\tau)} = \overline{s_2 s_1} + \overline{s_2^2}\tau = a + B\tau \tag{242}$$

$$\overline{s_3\Psi_b(\tau)} = \overline{s_1 s_3} + \overline{s_3^2}\ \tau^2 = b + C\ \tau^2 \quad , \tag{243}$$

where the factor $\frac{\Omega}{2}$ has been incorporated into s_3 for convenience and the substitutions are obvious. Substitution of equation 240 into 204 yields

$$A \int_o^t w_i(t,x)dx + B\tau \int_o^t x w_i(t,x)dx$$

$$+ C\tau^2 \int_o^t x^2 w_i(t,x)dx + a\left[\tau \int_o^t w_i(t,x)dx\right.$$

$$\left.\int_o^t x w_i(t,x)dx\right] + b\left[\tau^2 \int_o^t w_i(\ ,x)dx\right.$$

$$\left.+ \int_o^t x^2 w_i(t,x)dx\right] + G_o w_i(t,\tau) = \overline{s_i\Psi_b(\tau)}\ , \quad 0 \le \tau \le t \quad , \tag{244}$$

and again it can be seen that w_i has the form

$$w_i(t,\tau) = K_{i1}(t) + K_{i2}(t)\tau + K_{i3}(t)\tau^2 \quad . \tag{245}$$

Equation 244 then becomes

$$(A + a\tau + b\tau^2)(K_{i1}t + k_{i2}\frac{t^2}{2} + K_{i3}\frac{t^3}{3})$$

$$+ (B\tau + a)(K_{i1}\frac{t^2}{2} + K_{i2}\frac{t^3}{3} + K_{i3}\frac{t^4}{4})$$

$$+(C\tau^2 + b)(K_{i1}\frac{t^3}{3} + K_{i2}\frac{t^4}{4} + K_{i3}\frac{t^5}{5})$$

$$+ G_o(K_{i1} + K_{i2}\tau + K_{i3}\tau^2) = \overline{s_i\Psi_b(\tau)} \quad , \quad 0 \leq \tau \leq t \quad , \tag{246}$$

where the time dependence of the K's has been suppressed for compactness. Three equations for the K's are now written as before in matrix notation:

$$\underline{\underline{T}} \begin{pmatrix} K_{i1} \\ K_{i2} \\ K_{i3} \end{pmatrix} = \underline{V}_i \tag{247}$$

where the elements of $\underline{\underline{T}}$ are

$$T_{11} = G_o + At + a\frac{t^2}{2} + b\frac{t^3}{3} \tag{248}$$

$$T_{12} = A\frac{t^2}{2} + a\frac{t^3}{3} + b\frac{t^4}{4} \tag{249}$$

$$T_{13} = A\frac{t^3}{3} + a\frac{t^4}{4} + b\frac{t^5}{5} \tag{250}$$

$$T_{21} = at + B\frac{t^2}{2} \tag{251}$$

$$T_{22} = G_o + a\frac{t^2}{2} + B\frac{t^3}{3} \tag{252}$$

$$T_{23} = a\frac{t^3}{3} + B\frac{t^4}{4} \tag{253}$$

$$T_{31} = bt + C\frac{t^3}{3} \tag{254}$$

$$T_{32} = b\frac{t^2}{2} + C\frac{t^4}{4} \tag{255}$$

$$T_{33} = G_o + b\frac{t^3}{3} + C\frac{t^5}{5} \tag{256}$$

and the $\underline{V}_i$ vectors are

$$\underline{V}_1 = \begin{pmatrix} A \\ a \\ b \end{pmatrix} \tag{257}$$

$$\underline{V}_2 = \begin{pmatrix} a \\ B \\ o \end{pmatrix} \tag{258}$$

$$\underline{V}_3 = \begin{pmatrix} b \\ o \\ C \end{pmatrix} \tag{259}$$

It can be seen that this case is somewhat more complex than the uncorrelated case; however, equation 247 can certainly be solved for the K's in a straightforward manner. The resulting K's are similar in form to those found for the uncorrelated case and will not be written out explicitly, but in symbolic form:

$$\begin{pmatrix} K_{i1} \\ K_{i2} \\ K_{i3} \end{pmatrix} = \underline{\underline{T}}^{-1} \; \underline{V}_i$$

$$= \frac{1}{\Delta} \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} \underline{V}_i \quad , \tag{260}$$

where the C's are the cofactors of $\underline{\underline{T}}$. The determinant of $\underline{\underline{T}}$, Δ, is given by

$$\Delta = G_o{}^3 + G_o{}^2\, At + G_o{}^2\, at^2 + G_o{}^2\, (B + 2b)\frac{t^3}{3}$$

$$+ G_o(AB - a^2)\,\frac{t^4}{12} + \left(\frac{G_o{}^2 C}{5} - \frac{G_o ab}{6}\right)t^5$$

$$+ G_o\,\left(\frac{4}{45}(AC - b^2) - \frac{bB}{36}\right)t^6 + G_o\, aC\,\frac{t^7}{30}$$

$$+ G_o\, BC\,\frac{t^8}{240} + (ABC - Bb^2 - Ca^2)\frac{t^9}{2160} \quad . \tag{261}$$

Mean-square estimation errors will now be found using equations 229, 241, 242, 243 and 245. The results are (after some simplification)

$$\overline{e_{s1}^2(t)} = A = K_{11}(T_{11}-G_o) - K_{12}T_{12} - K_{13}T_{13} \tag{262}$$

$$\overline{e_{s2}^2(t)} = B - K_{21}T_{21} - K_{22}(T_{22}-G_o) - K_{23}T_{23} \tag{263}$$

$$\overline{e_{s3}^2(t)} = C - K_{31}T_{31} - K_{32}T_{32} - K_{33}(T_{33} - G_o) \quad . \tag{264}$$

The K's must now be evaluated symbolically through equations 257, 258, 259 and 260. The results, for $\overline{e_{s1}^2(t)}$, are

$$K_{11} = \frac{1}{\Delta}(C_{11}\, A + C_{21}\, a + C_{31}b) \tag{265}$$

$$K_{12} = \frac{1}{\Delta}(C_{12}\, A + C_{22}\, a + C_{32}b) \tag{266}$$

$$K_{13} = \frac{1}{\Delta}(C_{13}\, A + C_{23}\, a + C_{33}b) \quad . \tag{267}$$

Substitution of these equations into 262 and some rearrangement gives

$$\overline{e_{s1}^2(t)} = \frac{1}{\Delta}\{A\Delta - A(C_{11}T_{11} + C_{12}T_{12} + C_{13}T_{13})$$

$$-a(C_{21}T_{11} + C_{22}T_{12} + C_{23}T_{13})$$

$$-b(C_{31}T_{11} + C_{32}T_{12} + C_{33}T_{13})$$

$$+G_o(C_{11}\, A - C_{21}\, a + C_{31}\, b)\} \quad . \tag{268}$$

Now, it can be seen that the term $C_{11}T_{11} + C_{12}T_{12} + C_{13}T_{13}$ is just Δ, and that, by the so-called "alien cofactor rule", the terms $C_{21}T_{11} + C_{22}T_{12} + C_{23}T_{13}$ and $C_{31}T_{11} + C_{32}T_{12} + C_{33}T_{13}$ are both zero. (These terms are expansions for determinants with two identical rows.) Thus equation 268 reduces to

$$\overline{e_{s1}^2(t)} = \frac{G_o}{\Delta}(C_{11}\, A + C_{21}\, a + C_{31}\, b) \quad , \tag{269}$$

and, in expanded form, is

$$\overline{e_{s1}^2(t)} = \frac{G_o}{\Delta}\{G_o{}^2\, A + G_o(AB - a^2)\frac{t^3}{3}$$

$$- G_o\, ab\frac{t^4}{2} + G_o(AC - b^2)\frac{t^5}{5}$$

$$+ (ABC - Bb^2 - Ca^2)\frac{t^8}{240}\} \quad . \tag{270}$$

The errors $\overline{e_{s2}^2}$ and $\overline{e_{s3}^2}$ reduce, in exactly the same manner, to

$$\overline{e^2_{s2}(t)} = \frac{G_o}{\Delta}(C_{22}\ B + C_{12}\ a)$$

$$= \frac{G_o}{\Delta}\Bigg(G_o^{\ 2}\ B + G_o(AB - a^2)t$$

$$+ \frac{2}{3}\ G_o\ Bb\ t^3 + G_o\ BC\ \frac{t^5}{5}$$

$$+ \frac{4}{45}(ABC - Bb^2 - Ca^2)t^6\Bigg) \quad , \tag{271}$$

and

$$\overline{e^2_{s3}(t)} = \frac{G_o}{\Delta}(C_{33}\ C + C_{13}\ b)$$

$$= \frac{G_o}{\Delta}\ \{G_o^{\ 2}\ C + G_o(AC - b^2)t + G_o\ aC\ t^2$$

$$+ G_o\ BC\ \frac{t^3}{3} + (ABC - Bb^2 - Ca^2)\frac{t^4}{12}\} \quad . \tag{272}$$

It is comforting to note that these expressions reduce to those found earlier for the uncorrelated case when a and b are set equal to zero. The asymptotic expressions for these errors are identical to those for the uncorrelated case, equations 236, 237 and 238.

There are, of course, additional estimation errors present due to the approximations made in solving for the optimum filters. The most significant of these additional errors are those produced by the random character of gyro drift rates, since drift rates were approximated as constant. These errors are derived, using simplifying approximations, in Appendix B. Unfortunately, the significance of these errors can be determined only by numerical examples; two are given in the next Section.

The basic equations giving the signals available in the decoupled mode are equations 127 and 128. Inspection of these equations shows that $\Psi_W(t)$ has, as a coefficient, $\sin\eta$. Since the mean-square value of $\sin\eta$ is about

0.08, and noise terms of the same variance appear in both Ψ_u and Ψ_w, it was decided to use only $\Psi_{ub}(t)$ to estimate Ψ_{od} and $\Delta V'$. The derivation of the optimum filters and residual errors proceeds in exactly the same manner as in the normal operating mode cases. For simplicity, $\Psi_{ub}(t)$ is written

$$\Psi_{ub}(t) = f_1(t)\ \Psi_{od} + f_2(t)\ \Delta V' + n_u(t) \quad , \tag{273}$$

where the functions f_1 and f_2 represent the time functions in equation 127. The integral equation 204 is again applicable, where

$$\overline{\Psi_b(\tau)\Psi_b(x)} = f_1(\tau)\ f_1(x)\ \overline{\Psi^2_{od}} + f_2(\tau)\ f_2(x)\ \overline{\Delta V'^2}$$

$$+ G_o \delta(\tau - x) \quad , \tag{274}$$

$$\overline{s_1\Psi_b(\tau)} = \overline{\Psi_{od}\Psi_b(\tau)} = \overline{\Psi^2_{od}}\ f_1(\tau) \quad , \tag{275}$$

and

$$\overline{s_2\Psi_b(\tau)} = \overline{\Delta V'\Psi_b(\tau)} = \overline{\Delta V'^2}\ f_2(\tau) \quad . \tag{276}$$

The integral equation thus becomes

$$\begin{aligned} & f_1(\tau)\ \overline{\Psi^2_{od}} \int_o^t w_i(t,x)\ f_1(x)dx \\ & + f_2(\tau)\ \overline{\Delta V'^2} \int_o^t w_i(t,x)\ f_2(x)dx + G_o\ w_i(t,\tau) \\ & = \overline{s_i\Psi_b(\tau)} \quad , \quad 0 \le \tau \le t \ ; \end{aligned} \tag{277}$$

the weighting function clearly has the form

$$w_i(t,\tau) = K_{i1}(t)\ f_1(\tau) + K_{i2}(t)\ f_2(\tau) \quad ; \tag{278}$$

with this substitution, equation 277 becomes

$$\overline{s_i \Psi_b(\tau)} = \overline{\Psi^2_{od}}\, f_1(\tau)\, [K_{i1}(t) \int_o^t f_1{}^2(x)dx$$

$$+ K_{i2}(t) \int_o^t f_1(x)\, f_2(x)dx]$$

$$+ \overline{\Delta V'^2}\, f_2(\tau)\, [K_{i1}(t) \int_o^t f_1(x)\, f_2(x)dx$$

$$+ k_{i2}(t) \int_o^t f_2{}^2(x)dx] + G_o\, K_{i1}(t)\, f_1(\tau)$$

$$+ G_o\, K_{i2}(t)\, f_2(\tau), \quad 0 \leq \tau \leq t \quad ; \tag{279}$$

and finally, since f_1 and f_2 are linearly independent, the two equations for K_{i1} and K_{i2} can be written in matrix form from equation 279 as

$$\begin{pmatrix} (G_o + \overline{\Psi^2_{od}} \int_o^t f_1{}^2(x)dx) & (\overline{\Psi^2_{od}} \int_o^t f_1(x)\, f_2(x)dx) \\ (\overline{\Delta V'^2} \int_o^t f_1(x)\, f_2(x)dx) & (G_o + \overline{\Delta V'^2} \int_o^t f_2{}^2(x)dx) \end{pmatrix} \begin{pmatrix} K_{i1} \\ K_{i2} \end{pmatrix} = \underline{V}_i \ , \tag{280}$$

where

$$\underline{V}_1 = \begin{pmatrix} \overline{\Psi^2_{od}} \\ 0 \end{pmatrix} \tag{281}$$

and

$$\underline{V}_2 = \begin{pmatrix} 0 \\ \overline{\Delta V'^2} \end{pmatrix} \quad . \tag{282}$$

The K's can be found symbolically by first rewriting equation 280 as

$$\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} K_{i1} \\ K_{i2} \end{pmatrix} = \underline{V}_i \quad . \tag{283}$$

Then

$$K_{11} = \frac{\overline{\Psi_{od}^2}\, D_{22}}{\Delta} \tag{284}$$

$$K_{12} = -\frac{\overline{\Psi_{od}^2}\, D_{21}}{\Delta} \tag{285}$$

$$K_{21} = -\frac{\overline{\Delta V'^2}\, D_{12}}{\Delta} \tag{286}$$

$$K_{22} = \frac{\overline{\Delta V'^2}\, D_{11}}{\Delta} \tag{287}$$

where Δ is the determinant of the D matrix. The estimation errors are now found using equations 229, 275, 276 and 284 through 287. The results, after some simplifications, are

$$\overline{e^2_{\Psi od}} = \frac{G_o\, D_{22}\, \Psi^2_{od}}{\Delta} \tag{288}$$

and

$$\overline{e^2_{\Delta V'}} = \frac{G_o\, D_{11}\, \overline{\Delta V'^2}}{\Delta} \tag{289}$$

Although these equations are simple in form, the determinant Δ and the elements D_{11} and D_{22} are quite lengthy. They are therefore given in Appendix C, along with derivations of additional error terms due to neglecting drift rates and other components of $\underline{\Psi}$ in this mode.

Inspection of the form of the optimum filters derived shows that, even with all of the simplifying approximations made, they are still quite lengthy and would occupy a good part of the memory section of a typical computer. It can thus be at least conjectured that the filters obtained by the Kalman-Bucy or Shinbrot approaches for random drift rates, and even those obtained using both integral equations 148 and 149, would be considerably more lengthy and therefore not as feasible for mechanization. This is then the basic justification for these approximations and simplifications; the next Section gives two numerical examples showing that the filters derived explicitly here give adequate performance for the numerical values assumed.

VIII. NUMERICAL EXAMPLES SHOWING SYSTEM PERFORMANCE

As was stated in the Introduction, it is difficult to give numerical examples showing system performance because of security reasons. A few unclassified values that are available are estimates of random gyro drift rate correlation time and sun-tracker error parameters. It is known from experience that marine-type gyros exhibit drift rate autocorrelation functions of the form of equation 159, where $\frac{1}{\alpha}$ is of the order of 10 hours or more. It will therefore be assumed that $\frac{1}{\alpha}$ = 10 hours for these examples. Mansur (12) indicates radio-metric sun-trackers exhibit accuracy of the order of 30 seconds of arc, rms. To evaluate G_o, then, a sun-tracker error correlation time of one second was assumed, giving a triangular autocorrelation function with height 0.25 minutes^2 (of arc) and base two seconds. The coefficient G_o can then be found from the relation

$$\int_{-\infty}^{\infty} \overline{n(t)n(t+\tau)}\, d\tau = \int_{-\infty}^{\infty} G_o \delta(\tau) d\tau = G_o \quad . \tag{290}$$

Thus G_o represents the area under the autocorrelation function (and also the zero-frequency value of the power spectral density function $G(\omega)$, hence the notation), and has the value

$$G_o = \frac{1}{60}(0.25) = 0.004\ \widehat{\text{min}}^2\text{-min} \quad . \tag{291}$$

To save space and avoid confusion, angular units will be henceforth denoted by the "arc" sign ($\widehat{\text{min}}$,$\widehat{\text{sec}}$) and units of time by conventional abbreviations.

Random gyro drift rate variance ($\sigma_r{}^2$ in equation 159) will be postulated at a value of

$$\sigma_r{}^2 = 4\text{x}10^{-6}\ \widehat{\text{min}}^2/\text{min}^2 \quad . \tag{292}$$

This value is not meant to represent current capability, but was chosen because it represents a system which cannot operate in a damped inertial mode without checkpoints for more than a few days without developing errors of the order of several miles. Such a system could therefore benefit from the addition of a sun-tracker.

The uncorrelated case of Section VII is evaluated first. Here it is assumed that a system is activated at sea (perhaps after shut-down for repairs) and coarse initial alignment is accomplished. It is assumed that all components of $\underline{\Psi}_o$ have mean-square values

$$\overline{\Psi_o^{\,2}} = 5\ \widehat{\text{min}}^2 \quad , \tag{290}$$

and that gyro biases can be set to give drift rates for all gyros of

$$\overline{\varepsilon^2} = 10^{-4}\ \widehat{\text{min}}^2/\text{min}^2 \quad . \tag{291}$$

Evaluation of mean-square estimation errors is accomplished through equations 233, 234 and 235. The results are shown in Figures 10 and 11 as the dashed curves. Estimation errors are normalized to the initial values of the quantities; for example, $\dfrac{\overline{e^2_{\varepsilon e}(t)}}{\overline{\varepsilon_e^{\,2}}}$ is plotted in Figure 10. Additional errors due to random drift rates (equation 292) must be added to these; they are computed from the equations of Appendix B and give the solid curves of Figures 10 and 11 as the total estimation errors. It is interesting to note that minima are present on all curves except that for ε_e. This is not surprising in that the filters were derived for constant drift rates and are therefore not optimum for random drift rates. What apparently

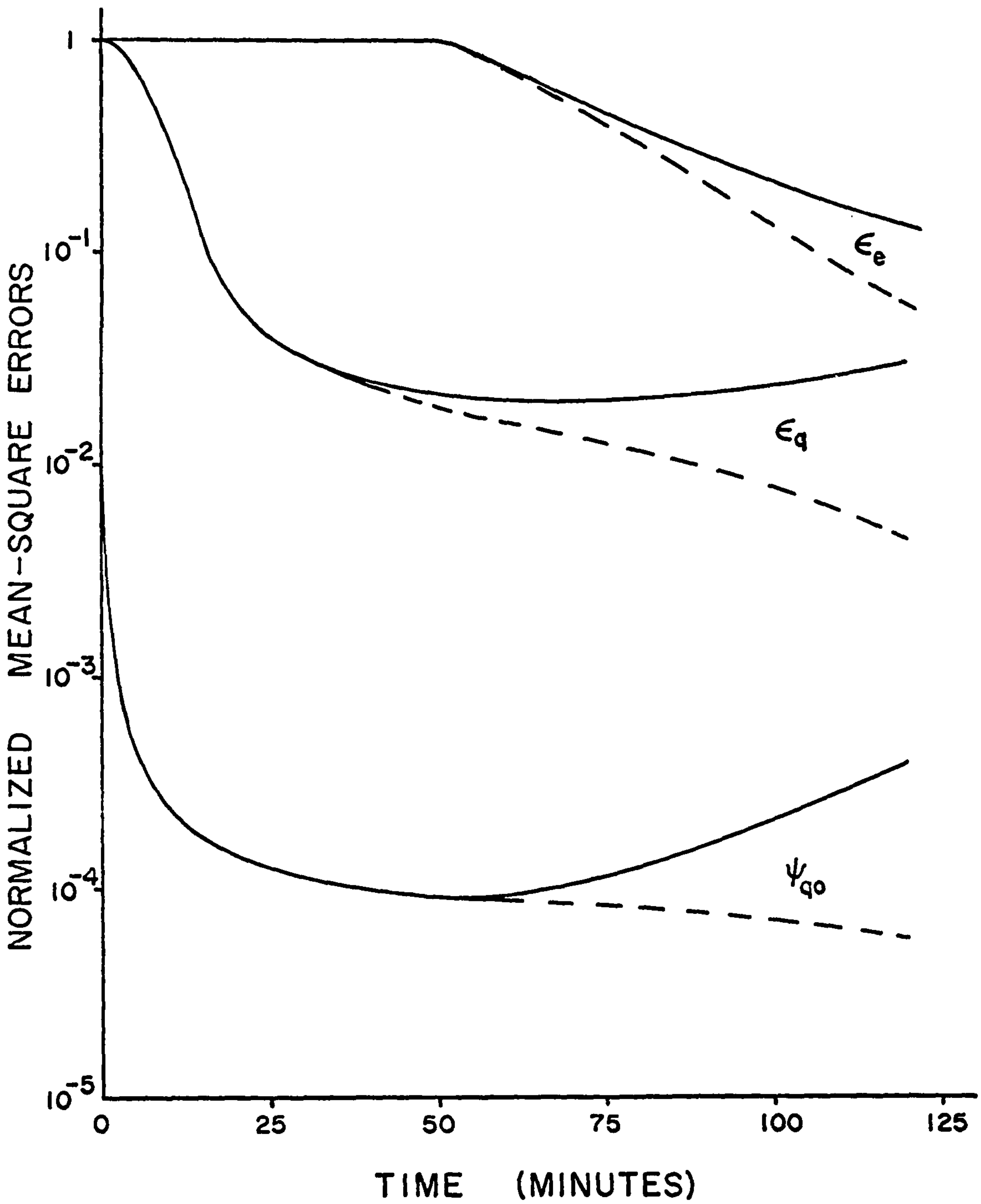

Figure 10. Normalized mean-square errors for uncorrelated case and Ψ_u

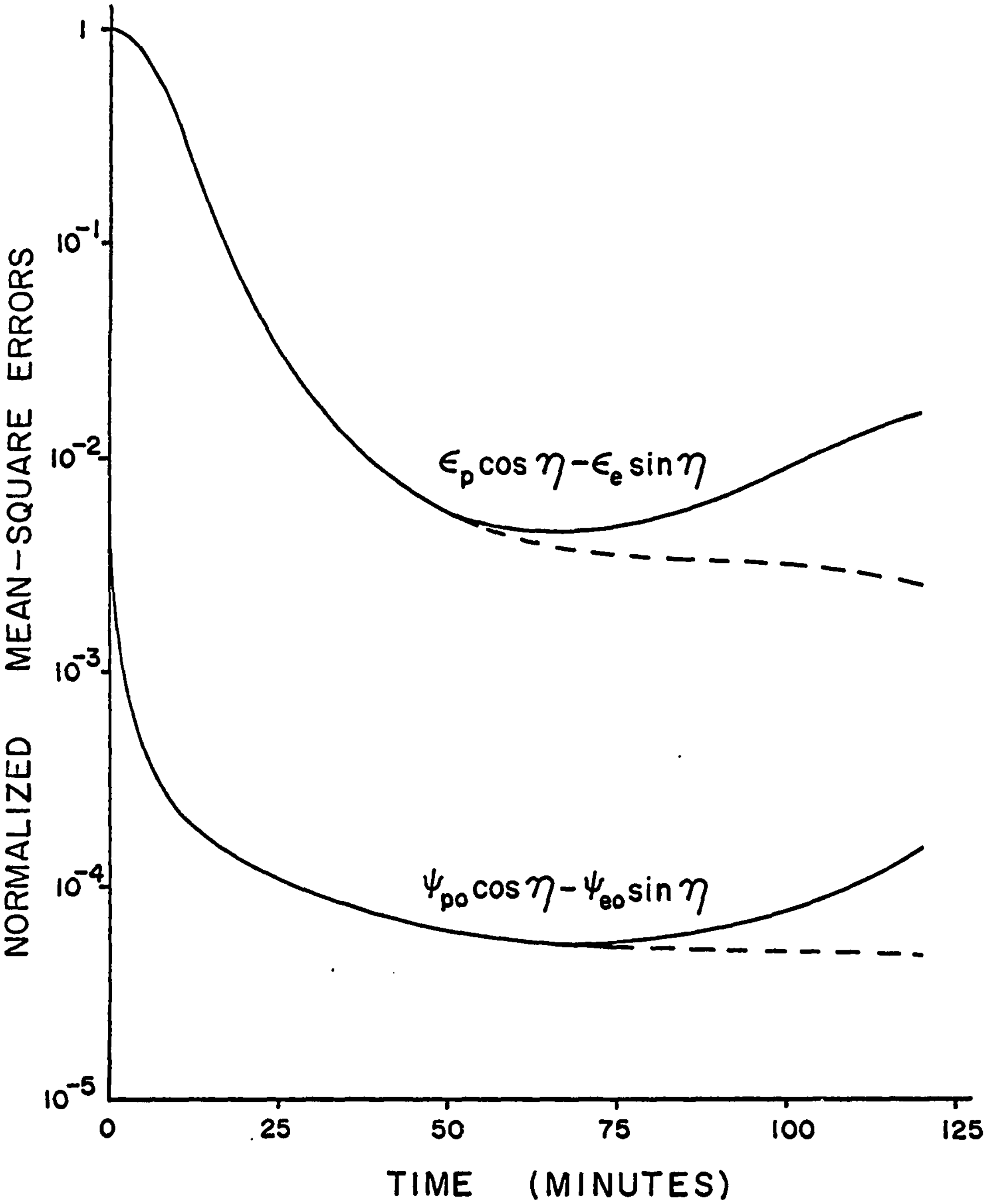

Figure 11. Normalized mean-square errors for uncorrelated case and Ψ_W

happens is that the filters give more weight to old data than a filter optimized for random drift rates would do; thus, as the present drift rate becomes uncorrelated with previous values more and more error is made. With this system, it would be a simple matter to use the estimate at the time of minimum error and ignore later estimates. The drift rate ε_p can be found from the equation

$$\varepsilon_p = \frac{1}{\cos\eta}[(\varepsilon_p \cos\eta - \varepsilon_e \sin\eta) + \varepsilon_e \sin\eta] \quad , \tag{292}$$

from which the mean-square estimation error in ε_p is

$$\overline{e^2_{\varepsilon_p}} = \frac{1}{\cos^2\eta}[\overline{(\varepsilon_p \cos\eta - \varepsilon_e \sin\eta)^2_E} + \overline{(\varepsilon_e \sin\eta)^2_E}] \quad , \tag{293}$$

where the E subscripts represent estimation errors. These values are read directly from Figures 10 and 11.

Estimation errors for the correlated or cyclic case must be found by iteration, since residual errors from a given day contribute to initial errors on the following day. Coupling is weak, however, and one iteration gives satisfactory results. It is assumed that the system operates in a damped inertial mode, without observations, for 20 hours (2 P.M. to 10 A.M.), in a cyclic, or day-to-day manner. Constant components of gyro drift rate will then be due to estimation errors. From estimates of drift rates and $\underline{\Psi}_o$, estimation errors are computed using the uncorrelated equations as an approximation. Initial values for the following day are then obtained from the equations of Appendix A. These values are dominated by gyro random drift rates; thus previous estimation errors are not too significant. The values obtained are:

$$\overline{\varepsilon_p^2(T)} = 7.3\text{x}10^{-6}\ \widehat{\text{min}}^2/\text{min}^2 \tag{294}$$

$$\overline{\Psi_p^{\,2}(T)} = 4.3\ \widehat{\min}^2 \tag{295}$$

$$\overline{\Psi_p(T)\varepsilon_p(T)} = 4.5\text{x}10^{-3}\ \widehat{\min}^2/\text{min} \tag{296}$$

$$\overline{\varepsilon_e^{\,2}(T)} = 10^{-5}\ \widehat{\min}^2/\text{min}^2 \tag{297}$$

$$\overline{\Psi_e^{\,2}(T)} = 0.64\ \widehat{\min}^2 \tag{298}$$

$$\overline{\varepsilon_q^{\,2}(T)} = 7.2\text{x}10^{-6}\ \widehat{\min}^2/\text{min}^2 \tag{299}$$

$$\overline{\Psi_q^{\,2}(T)} = 0.55\ \widehat{\min}^2 \tag{300}$$

$$\overline{\Psi_e(T)\varepsilon_e(T)} = -7.5\text{x}10^{-4}\ \widehat{\min}^2/\text{min} \tag{301}$$

$$\overline{\Psi_q(T)\varepsilon_q(T)} = -5\text{x}10^{-5}\ \widehat{\min}^2/\text{min} \tag{302}$$

$$\overline{\Psi_e(T)\varepsilon_q(T)} = -6.5\text{x}10^{-4}\ \widehat{\min}^2/\text{min} \tag{303}$$

$$\overline{\Psi_q(T)\varepsilon_e(T)} = 1.1\text{x}10^{-3}\ \widehat{\min}^2/\text{min} \tag{304}$$

These values, when substituted into equations 270, 271 and 272 give the estimation errors plotted as dashed curves in Figures 12 and 13. Again, the additional contributions due to random drift rates, when evaluated from Appendix B, give the solid curves as total estimation errors (normalized as before).

The behavior of the error in estimating ε_e (Figure 12) is at least curious at first glance. The initial decrease to about 0.85 and constant

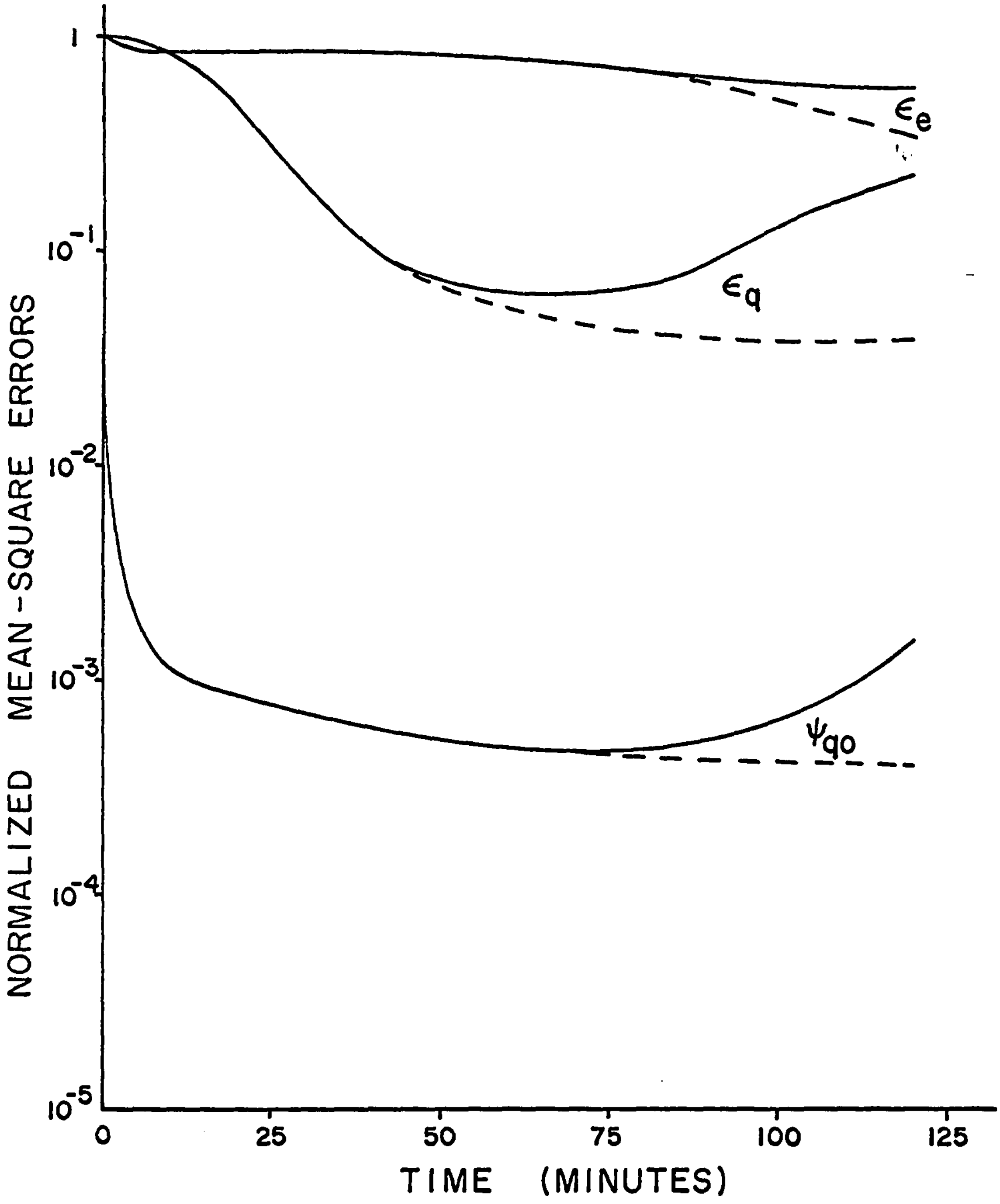

Figure 12. Normalized mean-square errors for cyclic case and Ψ_u

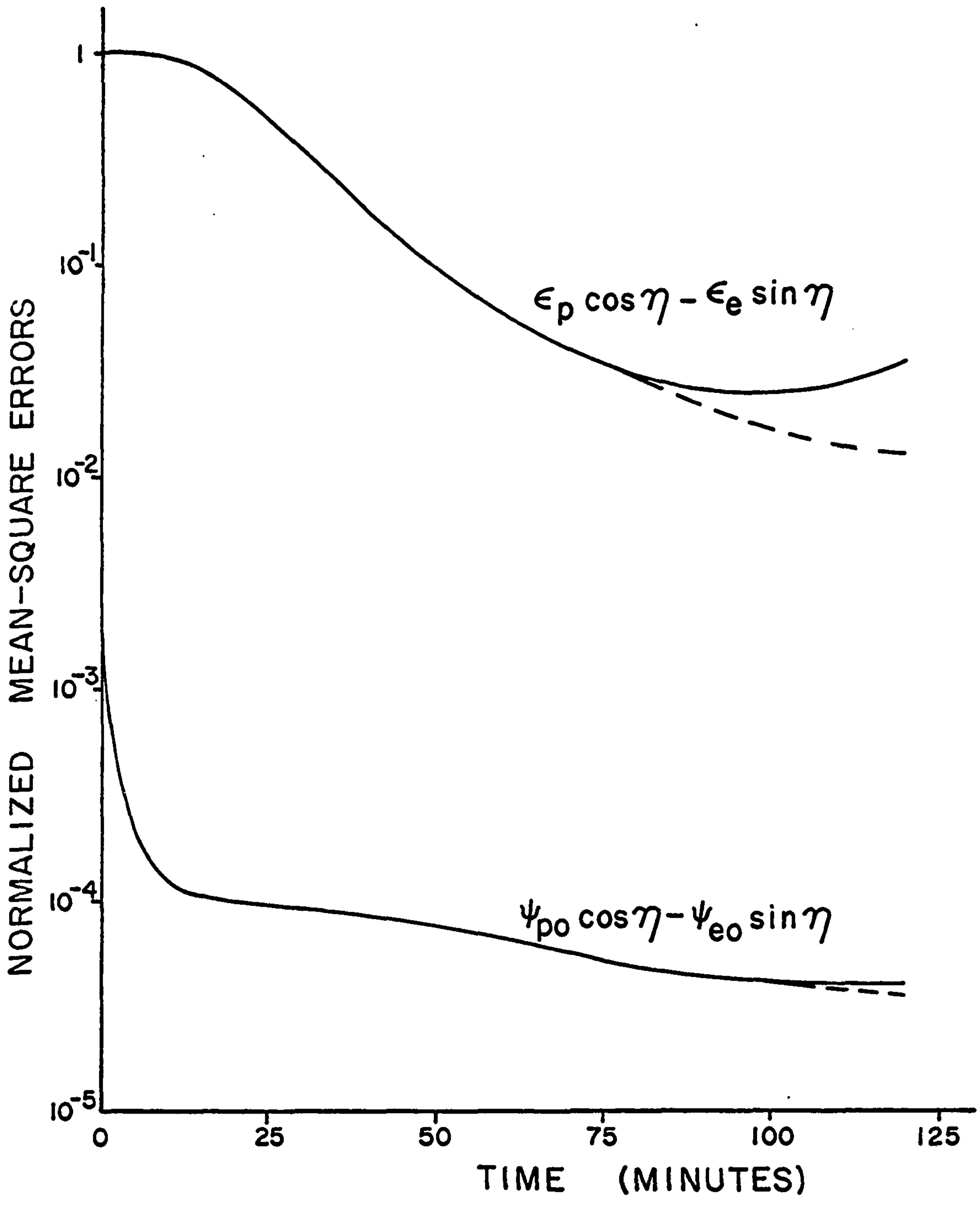

Figure 13. Normalized mean-square errors for cyclic case and Ψ_W

behavior until about one hour are easily explained, however, because of correlation between $\varepsilon_e(T)$ and $\Psi_{qo} = \Psi_q(T)$. Consider the estimation of a random variable X, when the only information available is the value of another random variable Y. If $\overline{XY} \neq 0$, a nontrivial optimum estimate can be obtained. If this estimate is linear, it can be written

$$Y_E = KX \quad . \tag{305}$$

The optimum value of K is found by forming the estimation error

$$E = Y - Y_E = Y - KX \quad , \tag{306}$$

and squaring and averaging E:

$$\overline{E^2} = \overline{Y^2} - 2\,K\,\overline{XY} + K^2\,\overline{X^2} \quad . \tag{307}$$

The optimum K is then found by differentiating $\overline{E^2}$ with respect to K:

$$\frac{\partial\,\overline{E^2}}{\partial\,K} = -\,2\,\overline{XY} + 2\,K\,\overline{X^2} = 0 \tag{308}$$

Thus

$$K_o = \frac{\overline{XY}}{\overline{X^2}} \quad , \tag{309}$$

The minimum error is then

$$\overline{E^2}_{min} = \overline{Y^2} - \frac{2(\overline{XY})^2}{\overline{X^2}} + \frac{(\overline{XY})^2}{\overline{X^2}} = \overline{Y^2} - \frac{(\overline{XY})^2}{\overline{X^2}} \quad , \tag{310}$$

and the normalized error is

$$\frac{\overline{E^2}_{min}}{\overline{Y^2}} = 1 - \frac{(\overline{XY})^2}{\overline{X^2}\;\overline{Y^2}} \quad . \tag{311}$$

For the particular case at hand,

$$\overline{X^2} = \overline{\Psi_{qo}^2} = 0.55 \ \widehat{\min}^2 \quad , \tag{312}$$

and, after 5 minutes, the error in estimating Ψ_{qo} is very small (Figure 12). The other quantities are

$$\overline{Y^2} = \frac{\Omega^2}{4} \ \overline{\varepsilon_e^{\ 2}} = 5\text{x}10^{-11} \ \widehat{\min}^2/\min^4 \tag{313}$$

and

$$(\overline{XY})^2 = (\frac{\Omega}{2} \ \overline{\Psi_{qo}\varepsilon_e})^2 = 4\text{x}10^{-12} \ \widehat{\min}^4/\min^4 \quad , \tag{314}$$

from equations 297, 300 and 304. The normalized estimation error is then, from equation 311,

$$\frac{\overline{E^2}_{\min}}{\overline{Y^2}} = 0.86 \quad , \tag{315}$$

which is in excellent agreement with the value from Figure 12. The filter, then, uses this principle to estimate ε_e (as it should).

Again, minima are present in all estimation error curves (except that for ε_e) because of the additional error due to random drift rates. In this case also, the estimation error for ε_q from Ψ_{wb} is not plotted because it is essentially useless. In the general case where two independent estimates of a random variable are available, the best unbiased estimate of the variable is given by

$$\varepsilon_{qE} = K \ \varepsilon_{qE1} + (1-K) \ \varepsilon_{qE2} \quad , \tag{316}$$

where

$$K = \frac{\overline{E_2^{\ 2}}}{\overline{E_1^{\ 2}}+\overline{E_2^{\ 2}}} \tag{317}$$

for minimum variance, and $\overline{E_i^2}$ is the mean-square estimation error for ε_{qEi}. The value $\overline{E_1^2}$, corresponding to the estimate of ε_q using Ψ_w, can be shown to be about $10^{-5}\ \widehat{\text{min}}^2/\text{min}^2$, and $\overline{E_2^2}$ is about $5\text{x}10^{-7}\ \widehat{\text{min}}^2/\text{min}^2$. Thus

$$K = 0.05 \quad , \tag{318}$$

and the error is

$$\overline{E^2} = \frac{\overline{E_1^2}\ \overline{E_2^2}}{\overline{E_1^2}+\overline{E_2^2}} = 0.95\ \overline{E_2^2} \tag{319}$$

Thus ε_{qE1} is of very little value.

In order to compute estimation errors for the decoupled modes, it is necessary to determine the system errors existing at the beginning of decoupled mode opeartion. These are found from Figures 10 through 13, and equations 201, 295 and 300. It is assumed that the normal operating mode is terminated after 100 minutes in the cyclic case, and after 120 minutes in the uncorrelated case. Initial errors for the cyclic case are then

$$\overline{\Psi_u^2(100)} = \overline{\Psi_{eo}^2} = 8\text{x}10^{-3}\ \widehat{\text{min}}^2 \tag{320}$$

$$\overline{\varepsilon_e^2} = \quad 2\text{x}10^{-6}\ \widehat{\text{min}}^2/\text{min}^2 \tag{321}$$

$$\overline{\varepsilon_q^2} = \quad 5\text{x}10^{-6}\ \widehat{\text{min}}^2/\text{min}^2 \tag{322}$$

$$\overline{\Psi_{od}^2} = 0.9\ \widehat{\text{min}}^2 \quad . \tag{323}$$

The uncorrelated case initial errors are

$$\overline{\Psi_u^{\;2}(100)} = \overline{\Psi_{eo}^2} = 5\text{x}10^{-2}\ \widehat{\text{min}}^2 \tag{324}$$

$$\overline{\varepsilon_e^{\;2}} = 6.5\text{x}10^{-6}\ \widehat{\text{min}}^2/\text{min}^2 \tag{325}$$

$$\overline{\varepsilon_q^{\;2}} = 0.5\text{x}10^{-5}\ \widehat{\text{min}}^2/\text{min}^2 \tag{326}$$

$$\overline{\Psi_{od}^2} = 5\ \widehat{\text{min}}^2 \quad . \tag{327}$$

In both cases the values of $\overline{\varepsilon_e^{\;2}}$ and $\overline{\varepsilon_q^{\;2}}$ are resolved to the epq coordinate system from the e'pq' system used in the normal operating mode. A value of ω_1 corresponding to a two-hour period was chosen and a mean-square reference velocity error of 9 ft^2/sec^2 was assumed. Thus the numerical value of ω_1 is

$$\omega_1 = 0.052/\text{min} \quad , \tag{328}$$

and from equation 112,

$$K = 143\ \Omega = 0.61/\text{min} \quad . \tag{329}$$

Then, from equation 113, $\overline{\Delta V'^2}$ is

$$\overline{\Delta V'^2} = 2.8\text{x}10^{-11}/\text{min}^4 \quad . \tag{330}$$

The observation period here was chosen as two hours. The values given in equations 320 through 330, when substituted into the equations of Appendix C and equations 288 and 289 give the following errors after two hours:

Uncorrelated case

$$\overline{e_{\Psi od}^2} = 1.6\text{x}10^{-1}\ \widehat{\text{min}}^2 \tag{331}$$

$$\overline{e^2_{\frac{\Delta V'}{\omega_1^2}}} = 5.8x10^{-3}\ \widehat{\text{min}}^2 \tag{332}$$

Cyclic case

$$\overline{e^2_{\Psi ai}} = 6.3x10^{-2}\ \widehat{\text{min}}^2 \tag{333}$$

$$\overline{e^2_{\frac{\Delta V'}{\omega_1^2}}} = 1.7x10^{-3}\ \widehat{\text{min}}^2 \quad , \tag{334}$$

where the errors in estimating $\frac{\Delta V'}{\omega_1^2}$ are given, as they are more easily compared with other errors. In both cases, ε_e is the dominant contributing error.

Table 1 gives a summary of root-mean-square (rms) estimation errors for both cases.

Table 1. RMS estimation errors

Estimated quantity	RMS error, uncorrelated case	RMS error, cyclic case
ε_e	0.0036 $\widehat{\text{min}}$/min	0.0024 $\widehat{\text{min}}$/min
ε_p	0.0015 $\widehat{\text{min}}$/min	0.0009 $\widehat{\text{min}}$/min
ε_q	0.0015 $\widehat{\text{min}}$/min	0.0007 $\widehat{\text{min}}$/min
Ψ_{qo}	0.021 $\widehat{\text{min}}$	0.016 $\widehat{\text{min}}$
$\Psi_{po} \cos \eta - \Psi_{eo} \sin \eta$	0.016 $\widehat{\text{min}}$	0.013 $\widehat{\text{min}}$
$\Psi_{po} \sin \eta + \Psi_{eo} \cos \eta$	0.4 $\widehat{\text{min}}$	0.25 $\widehat{\text{min}}$
$\frac{\Delta V'}{\omega_1^2}$	0.076 $\widehat{\text{min}}$	0.041 $\widehat{\text{min}}$

The equations of Appendix B can be used to compute continuous rms errors for the period of time between observations, and equations 68, 69 and 70 to give latitude, longitude and azimuth errors due to $\underline{\Psi}$. Such values would have real significance, however, only when drift rate variances and other error values from an actual system are used; they are therefore not plotted here.

IX. CONCLUSIONS

It is clear from the material presented that the mechanization derived here has succeeded in bounding mean-square system errors; this was the primary intent of this study. The mechanization is also clearly not the absolute optimum for the general case of random gyro drift rates. It is the optimum linear mechanization for constant drift rates, however, and its form indicates that the true optimum mechanization could probably not be accomodated within a typical computer together with the conventional mechanization equations required for such a system. This, then, is the justification for using this mechanization, and it is regarded as optimum under this constraint.

It is unfortunate that security reasons preclude an evaluation of this system using error quantities from an actual system or components. The numerical examples given, however, provide some insight into this aspect of the problem, and indicate that, for these values, the mechanization is certainly worthwhile. The cyclic example provides some further information, in that the estimation error for ε_e has essentially the same rms value as the random drift rate. Thus, for smaller rms drift rates, it appears that a higher accuracy tracker would be very worthwhile. The error values used for random drift rates and tracker accuracy seem, then, to be compatible in that reduction of either one alone would not achieve a proportionate increase in over-all system accuracy.

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XI. ACKNOWLEDGEMENT

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XII. APPENDIX A

In this section are derived equations giving mean-square errors and cross-correlation terms for $\underline{\Psi}$ propagation in a damped inertial mode. It is assumed that damped inertial operation commences at t=0, and observations commence at t=T. The quantities desired, then are $\overline{\varepsilon_p^{\,2}(T)}$, $\overline{\varepsilon_e^{\,2}(t)}$, $\overline{\varepsilon_q^{\,2}(T)}$, $\overline{\Psi_p^{\,2}(T)}$, $\overline{\Psi_e^{\,2}(T)}$, $\overline{\Psi_q^{\,2}(T)}$, $\overline{\Psi_e(T)\ \varepsilon_e(T)}$, $\overline{\Psi_e(T)\ \varepsilon_q(T)}$, $\overline{\Psi_q(T)\ \varepsilon_e(T)}$, $\overline{\Psi_p(T)\ \varepsilon_p(T)}$ and $\overline{\Psi_q(T)\ \varepsilon_q(T)}$. (For convenience, the "primed" notation has been dropped.) These quantities are derived from equations 60, 159 and 239.

The equations for drift rate variance are the simplest and will be derived first. Their general form is found by squaring and averaging equation 239:

$$\overline{\varepsilon^2(T)} = \overline{\varepsilon_r^{\,2}(T)} + \overline{\varepsilon_r^{\,2}(o)} + \overline{\varepsilon_E^{\,2}} - 2\ \overline{\varepsilon_r(T)\varepsilon_r(o)} \quad , \tag{335}$$

since other cross-product terms are zero. Substitution of equation 159 into 335 gives

$$\overline{\varepsilon^2(T)} = \sigma_E^{\,2} + 2\ \sigma_r^{\,2}\ (1 - e^{-\alpha T}) \quad , \tag{336}$$

where $\sigma_E^{\,2} = \overline{\varepsilon_E^{\,2}}$. Addition of appropriate subscripts to $\overline{\sigma_E^{\,2}}$ then gives the three drift rate variances. Drift rate autocorrelation functions are also required, and are found in the same manner:

$$\overline{\varepsilon(u)\varepsilon(v)} = \overline{\varepsilon_r(u)\varepsilon_r(v)} + \overline{\varepsilon_r^{\,2}(o)} + \overline{\varepsilon_E^{\,2}} - \overline{\varepsilon_r(u)\varepsilon_r(o)} - \overline{\varepsilon_r(v)\varepsilon_r(o)}$$

$$= \sigma_E^{\,2} + \sigma_r^{\,2}[e^{-\alpha|u-v|} - e^{-\alpha u} - e^{-\alpha v} + 1]\ ,\ u,v \geq 0. \tag{337}$$

Again, appropriate subscripts are added to $\sigma_E^{\,2}$ to give values for individual gyros.

Next $\Psi_p^2(T)$ is found from equations 60 by squaring and averaging:

$$\overline{\Psi_p^{\;2}(T)} = \overline{\Psi_{po}^2} + \int_o^T \int_o^T \overline{\varepsilon_p(u)\varepsilon_p(v)}\, du\, dv \quad . \tag{338}$$

From equation 337 and by the symmetry of the double integral, equation 338 can be rewritten

$$\begin{aligned}\overline{\Psi_p^{\;2}(T)} &= \overline{\Psi_{po}^2} + (\sigma_r^{\;2} + \sigma_E^{\;2})T^2 - 2\,\sigma_E^{\;2}\,\frac{T}{\alpha}\,(1 - e^{-\alpha T}) \\ &+ 2\int_o^T \int_o^u \sigma_r^{\;2}\, e^{-\alpha(u-v)}\, dv\, du \\ &= \overline{\Psi_{po}^2} + (\sigma_r^{\;2} + \sigma_E^{\;2})T^2 + 2\,\sigma_r^{\;2}\,\frac{T}{\alpha} - \frac{2\,\sigma_r^{\;2}}{\alpha^2}\,(T+\alpha)(1-e^{-\alpha T}) \quad .\end{aligned} \tag{339}$$

The cross-correlation term $\overline{\Psi_p(T)\varepsilon_p(T)}$ is found from equations 60, 159, 239 and 337:

$$\begin{aligned}\overline{\Psi_p(T)\varepsilon_p(T)} &= \int_o^T \overline{\varepsilon_p(T)\varepsilon_p(u)}\, du \\ &= \int_o^T [\sigma_r^{\;2}(e^{-\alpha(T-u)} - e^{-\alpha T} - e^{-\alpha u} + 1) + \overline{\sigma_{Ep}^2}]\, du \\ &= \sigma_r^{\;2}\, T\,(1 - e^{-\alpha T}) + \sigma_{Ep}^2\, T \quad .\end{aligned} \tag{340}$$

The remaining terms have similar forms, so it is not necessary to derive all of them explicitly. It will be seen that derivations of $\overline{\Psi_e^{\;2}(T)}$, $\overline{\Psi_e(T)\varepsilon_e(T)}$ and $\overline{\Psi_e(T)\varepsilon_q(T)}$ will be sufficient. From equation 60, $\overline{\Psi_e^{\;2}(T)}$ can be written

$$\begin{aligned}\overline{\Psi_e^{\;2}(T)} &= \overline{\Psi_{eo}^2}\cos^2\Omega T + \overline{\Psi_{qo}^2}\sin^2\Omega T + \int_o^T \int_o^T \Big\{\cos\Omega u\,\cos\Omega v \\ &(\overline{\varepsilon_e(T-u)\varepsilon_e(T-v)}) + \sin\Omega u\,\sin\Omega v(\overline{\varepsilon_q(T-u)\varepsilon_q(T-v)})\, du\, dv\Big\} \quad .\end{aligned} \tag{341}$$

Use of equation 337 and a trigonometric identity results in

$$\overline{\Psi_e^{\,2}(T)} = \overline{\Psi_{eo}^2}\cos^2\Omega T + \overline{\Psi_{qo}^2}\sin^2\Omega T$$

$$+ \int_o^T\int_o^T [\sigma_r^{\,2}\cos\Omega(u-v)(e^{-\alpha|u-v|} - 2\,e^{-\alpha(T-u)} + 1)$$

$$+ \cos\Omega u\cos\Omega v\,\sigma_{Ee}^2 + \sin\Omega u\sin\Omega v\,\sigma_{Eq}^2]du\,dv$$

$$= \overline{\Psi_{eo}^2}\cos^2\Omega T + \overline{\Psi_{qo}^2}\sin^2\Omega T + \sigma_{Ee}^2\frac{\sin^2\Omega T}{\Omega^2} + \sigma_{Eq}^2\left(\frac{1-\cos\Omega T}{\Omega}\right)^2$$

$$+ 2\int_o^T\int_o^v \sigma_r^{\,2}\cos\Omega(u-v)(e^{-\alpha(v-u)} - 2\,e^{-\alpha(T-u)} + 1)\,du\,dv \quad , (342)$$

where the symmetry of the remaining integral has been utilized to eliminate the absolute value signs. Complete evaluation of equation 342 involves only standard integration; the result is

$$\overline{\Psi_e^{\,2}(T)} = \overline{\Psi_{eo}^2}\cos^2\Omega T + \overline{\Psi_{qo}^2}\sin^2\Omega T + \sigma_{Ee}^2\frac{\sin^2\Omega T}{\Omega^2} + \sigma_{Eq}^2\left(\frac{1-\cos\Omega T}{\Omega}\right)^2$$

$$+ 2\,\sigma_r^{\,2}\,\{\frac{1-\cos\Omega T}{\Omega^2} + \frac{1}{\alpha^2+\Omega^2}(\alpha T + 2e^{-\alpha T}(\frac{\alpha}{\Omega}\sin\Omega T + \cos\Omega T) - 2)$$

$$+ \frac{1}{(\alpha^2+\Omega^2)^2}((\alpha^2-\Omega^2)\,e^{-\alpha T}\cos\Omega T - 2\alpha\Omega\,e^{-\alpha T}\sin\Omega T + \Omega^2-\alpha^2)\} \,. \quad (343)$$

Inspection of equations 60 shows that $\overline{\Psi_q^{\,2}(T)}$ can be obtained from equation 343 by merely interchanging the e and q subscripts. (The differences in sign are lost in the squaring process.)

The next term to be found is $\overline{\Psi_e(T)\varepsilon_e(T)}$. This term is written from equations 60, 159 and 337:

$$\overline{\Psi_e(t)\varepsilon_e(T)} = \int_o^T \cos\Omega u\;(\overline{\varepsilon_e(T)\varepsilon_e(T-u)})\;du$$

$$= \int_0^T \cos\Omega u[\sigma_r^{\,2}(e^{-\alpha u} - e^{-\alpha T} - e^{-\alpha(T-u)} + 1) + \sigma_{Ee}^2]du \quad . \tag{344}$$

This is a straightforward integration; the result is

$$\overline{\Psi_e(T)\varepsilon_e(T)} = [\sigma_r^{\,2}(1 - e^{-\alpha T}) + \sigma_{Ee}^2]\,\frac{\sin\Omega T}{\Omega}$$

$$+ \frac{\sigma_r^{\,2}}{\alpha^2+\Omega^2}[e^{-\alpha T}(-\alpha\cos\Omega T + \Omega\sin\Omega T + \alpha)$$

$$+ \alpha(1-\cos\Omega T) - \Omega\sin\Omega T] \quad . \tag{345}$$

It can be seen that $\overline{\Psi_q(T)\varepsilon_q(T)}$ is obtained from equation 345 merely by substituting σ_{Eq}^2 for σ_{Ee}^2.

The term $\overline{\Psi_e(T)\varepsilon_q(T)}$, from equations 60, 159 and 337, is:

$$\overline{\Psi_e(T)\varepsilon_q(T)} = -\int_0^T \sin\Omega u[\sigma_r^{\,2}(e^{-\alpha u} - e^{-\alpha T} - e^{-\alpha(T-u)} + 1) + \sigma_{Eq}^2]du$$

$$= -[\sigma_r^{\,2}(1 - e^{-\alpha T}) + \sigma_{Eq}^2]\left(\frac{1-\cos\Omega T}{\Omega}\right) - \frac{\sigma_r^{\,2}}{\alpha^2+\Omega^2}$$

$$[e^{-\alpha T}(\Omega(1-\cos\Omega T) - \alpha\sin\Omega T) + \Omega(1-\cos\Omega T) + \alpha\sin\Omega T] \quad . \tag{346}$$

The term $\overline{\Psi_q(T)\varepsilon_e(T)}$ is obtained from equation 346 by changing the algebraic sign and substituting e for q.

These equations are used in the numerical examples of Section VIII. They are of some general interest in that they show that $\underline{\Psi}$ errors grow with time in an unbounded manner, as was stated earlier. This is illustrated by equation 343, where, for large T, $\overline{\Psi_e^{\,2}(T)}$ is proportional to T as in the classical random walk process.

XIII. APPENDIX B

Additional errors which occur during filtering due to random gyro drift rates are derived here. These equations are derived only for the uncorrelated, normal operating mode case. Since the filters for the correlated or cyclic case asymptotically approach the same forms as those in the first case, it is a reasonable approximation to use these results in both cases.

In order to find these additional errors, gyro drift rates during an observation interval will be written

$$\varepsilon(t) = \varepsilon_c + \varepsilon_r(t) \quad , \tag{347}$$

where ε_c represents the constant component and $\varepsilon_r(t)$ the random component. The components of $\underline{\Psi}$ can then be written

$$\Psi_p(t) = \Psi_{po} + \varepsilon_{pc} t + \int_0^t \varepsilon_{pr}(\tau)\, d\tau \tag{348}$$

$$\begin{aligned} \Psi_e(t) &= \Psi_{eo} \cos\Omega t - \Psi_{qo} \sin\Omega t \\ &+ \varepsilon_{ec} \frac{\sin\Omega t}{\Omega} + \varepsilon_{qc}\left(\frac{\cos\Omega t - 1}{\Omega}\right) \\ &+ \int_0^t [\cos\Omega(t-\tau)\varepsilon_{er}(\tau) - \sin\Omega(t-\tau)\varepsilon_{qr}(\tau)]d\tau \end{aligned} \tag{349}$$

$$\begin{aligned} \Psi_q(t) &= \Psi_{qo} \cos\Omega t + \Psi_{eo} \sin\Omega t \\ &+ \varepsilon_{qc} \frac{\sin\Omega t}{\Omega} + \varepsilon_{ec}\left(\frac{1-\cos\Omega t}{\Omega}\right) \\ &+ \int_0^t [\cos\Omega(t-\tau)\varepsilon_{qr}(\tau) + \sin\Omega(t-\tau)\varepsilon_{er}(\tau)]d\tau \quad . \end{aligned} \tag{350}$$

Expansion of the cosine and sine terms in equations 349 and 350 and use of equations 78 and 79 gives

$$\Psi_{ub}(t) = \Psi_{qo} + \varepsilon_{qc}\frac{\sin\Omega t}{\Omega} + \varepsilon_{ec}\left(\frac{\cos\Omega t-1}{\Omega}\right)$$

$$+ \int_0^t [\cos\Omega\tau\ \varepsilon_{qr}(\tau) - \sin\Omega\tau\ \varepsilon_{er}(\tau)]d\tau + n_u(t) \quad , \tag{351}$$

$$\Psi_w(t) = (\Psi_{po}\cos\eta - \Psi_{eo}\sin\eta) + \varepsilon_p\cos\eta t$$

$$-\varepsilon_e\sin\eta\ \frac{\sin\Omega t}{\Omega} + \varepsilon_q\frac{\sin\eta}{\Omega}(\cos\Omega t-1)$$

$$+ \cos\eta\int_0^t \varepsilon_{pr}(\tau)d\tau - \sin\eta\int_0^t[\sin\Omega\tau\ \varepsilon_{qr}(\tau)$$

$$+ \cos\Omega\tau\ \varepsilon_{er}(\tau)]d\tau + n_w(t) \quad . \tag{352}$$

All terms in these equations not containing integrals are the values of Ψ_u and Ψ_w previously used; these will be denoted here by Ψ_{uc} and Ψ_{wc}. Then equations 351 and 352 can be written

$$\Psi_{ub}(t) = \Psi_{uc}(t) + \Psi_{ur}(t) \quad , \tag{353}$$

and

$$\Psi_{wb}(t) = \Psi_{wc}(t) + \Psi_{wr}(t) \quad , \tag{354}$$

where Ψ_{ur} and Ψ_{wr} are the terms due to random drift rates. (The primed notation previously used has also been dropped here.)

The first additional error to be found is that in the estimate of Ψ_{qo}. The mean-square error is given by equation 228:

$$\overline{e^2_{\Psi_{qo}}(t)} = \overline{\Psi^2_{qo}} - 2\int_0^t \overline{\Psi_{qo}\Psi_{ub}(\tau)}\ w_1(t,\tau)d\tau$$

$$+ \int_0^t\int_0^t \overline{\Psi_{ub}(\tau)\Psi_{wb}(v)}\ w_1(t,\tau)\ w_1(t,v)d\tau\ dv \quad , \tag{355}$$

which reduces to

$$\overline{e^2_{\Psi_{qo}}(t)} = \overline{\Psi^2_{qo}} - \int_o^t \overline{\Psi_{qo}\Psi_{uc}(\tau)}\, w_1(t,\tau)d\tau$$

$$+ \int_o^t\int_o^t (\overline{\Psi_{uc}(\tau)\Psi_{uc}(v)} + \overline{\Psi_{ur}(\tau)\Psi_{ur}(v)})\, w_1(t,\tau)\, w_1(t,v)d\tau\, dv \quad , (356)$$

where correlation between Ψ_{uc} and Ψ_{ur} is ignored as being small and Ψ_{qo} is clearly not, in this case, correlated with Ψ_{ur}. The new term to be evaluated, then, is

$$\overline{e^2_{r_1}(t)} = \int_o^t\int_o^t (\overline{\Psi_{ur}(\tau)\Psi_{ur}(v)})\, w_1(t,\tau)\, w_1(t,v)d\tau\, dv$$

$$= 2\int_{x=o}^t\int_{\tau=o}^v \overline{\Psi_{ur}(\tau)\Psi_{ur}(v)}\, w_1(t,\tau)\, w_1(t,v)d\tau\, dv \qquad (357)$$

by symmetry of the integral. From equation 351, the term $\overline{\Psi_{ur}(\tau)\Psi_{ur}(v)}$ can be written

$$\overline{\Psi_{ur}(\tau)\Psi_{ur}(v)} = \int_o^\tau\int_o^v [\cos\Omega(x-y)(\sigma_r{}^2 e^{-\alpha|x-y|})]dx\, dy \qquad (358)$$

For $\tau \leq v$ (the condition in equation 357), equation 358 can be written

$$\overline{\Psi_{ur}(\tau)\Psi_{ur}(v)} = 2\,\sigma_r{}^2 \int_o^\tau (\int_o^y \cos\Omega(x-y)e^{-\alpha(y-x)}dx)dy$$

$$+ \sigma_r{}^2 \int_o^\tau\int_\tau^v \cos\Omega(x-y)e^{-\alpha(x-y)}dx)dy \qquad (359)$$

again using the symmetry of the integral.

Now, two simplifying approximations are introduced. The first is to replace $\cos\Omega(x-y)$ by $1 + \Omega^2 xy$; this is equivalent to the approximations made earlier for $\sin\Omega t$ and $\cos\Omega t$, as will be seen. The second is to use the first-order term in the expansion of $e^{-\alpha t}$: $e^{-\alpha(y-x)} = 1 - \alpha(y-x)$.

Since observation time is expected to be one or two hours and $\frac{1}{\alpha} \doteq 10$ hours, this is quite accurate. With these approximations equation 359 becomes

$$\overline{\Psi_{ur}(\tau)\Psi_{ur}(v)} \doteq \sigma_r^{\ 2} \{2 \int_o^\tau \int_o^y (1+\Omega^2 xy)(1+xx-xy)dx)dy$$
$$+ \int_o^\tau (\int_\tau^v (1+\Omega^2 xy)(1+\alpha y-\alpha x)dx)dy \quad . \tag{360}$$

These integrations are quite straightforward; the result is

$$\overline{\Psi_{ur}(\tau)\Psi_{ur}(v)} \doteq \sigma_r^{\ 2}[v\tau + \frac{\Omega^2 v^2 \tau^2}{4} + \alpha(\frac{v\tau^2}{2} - \frac{\tau v^2}{2} - \frac{\tau^3}{3})$$
$$+ \alpha\Omega^2(\frac{v^2\tau^3}{6} - \frac{v^3\tau^2}{6} - \frac{\tau^5}{15})] \quad . \tag{361}$$

It is now observed that the first two terms in this expression are identical to those in equation 205 due to gyro drift rates. This is to be expected, since these terms are the "zero-order" terms in an expansion of $\overline{\Psi_{ur}(\tau)\Psi_{ur}(v)}$ with respect to α; if α went to zero, the random drift rates would become constant and these two terms would be the only ones present. The additional error introduced in the estimate of Ψ_{qo}, then, is due to the remaining terms in equation 361, since drift rate variances used in constructing the optimum filters will be total variances, including $\sigma_r^{\ 2}$.

The additional error, then, is given, to a good approximation, by

$$\overline{e_r^{\ 2}} = 2\, \sigma_r^{\ 2}\alpha \int_o^t [\int_o^v w_1(t,\tau)\, w_1(t,v) \{\frac{v\tau^2}{2} - \frac{\tau v^2}{2} - \frac{\tau^3}{3}$$
$$+ \Omega^2(\frac{v^2\tau^3}{6} - \frac{v^3\tau^2}{6} - \frac{\tau^5}{15})\}\, d\tau]dv \tag{362}$$

Substitution of equation 211 for w_1 into equation 362 gives, after integrating and collecting terms:

$$\overline{e_r^{\,2}} = -\,\sigma_r^{\,2}\alpha t^5\Bigg\{\frac{K_{11}^{\,2}}{15} + 11\,\frac{K_{11}K_{12}t}{120}$$

$$+ \frac{89\,K_{11}K_{13}t^2}{1260} + \frac{13\,K_{12}^{\,2}t^2}{420} + \frac{17\,K_{12}K_{13}t^3}{360}$$

$$+ \frac{29\,K_{13}^{\,2}t^4}{1620} + \Omega^2t^2\Bigg(\frac{K_{11}^{\,2}}{140} + \frac{3\,K_{11}K_{12}t}{280}$$

$$+ \frac{K_{12}^{\,2}t^2}{252} + \frac{7\,K_{11}K_{13}t^2}{810} + \frac{2\,K_{12}K_{13}t^3}{315}$$

$$+ \frac{K_{13}^{\,2}t^4}{396}\Bigg)\Bigg\} \tag{363}$$

It can be seen that the approximations made for $\cos\Omega(x-y)$ and $e^{-\alpha|x-y|}$ were certainly in order, if only to keep the expression for $\overline{e_r^{\,2}}$ from becoming completely intractable.

A similar analysis for the additional error in $\Psi_{wo} = \Psi_{po}\cos\eta - \Psi_{eo}\sin\eta$ shows that equation 363, with Ω^2 replaced by $\Omega^2\sin^2\eta$, gives the error in Ψ_{wo}.

The additional error in estimating $\varepsilon_q(t)$ is found in the same manner. The total error is written from equation 228:

$$\overline{e^2_{\varepsilon q}(t)} = \overline{\varepsilon_q^{\,2}(t)} - 2\int_0^t \overline{\varepsilon_q(t)\Psi_{ub}(\tau)}\; w_2(t,\tau)\;d\tau$$

$$+ \int_0^t\int_0^t \overline{\Psi_{ub}(\tau)\Psi_{ub}(v)}\; w_2(t,\tau)\; w_2(t,v)\; d\tau\; dv \quad . \tag{364}$$

The third term in equation 364 is expanded exactly as before and results in a component of additional error given by equation 362, with the K's changed to those used in the filter for estimating ε_q. This term will be denoted $\overline{e_{r12}^2(t)}$. There is another term present, which arises from the second term in equation 364. The covariance function in this term can be written, using equation 347 and 351, as

$$\overline{\varepsilon_q(t)\Psi_{ub}(\tau)} = \overline{\varepsilon_{qc}^2} \; \frac{\sin\Omega\tau}{\Omega}$$

$$+ \int_0^\tau \sigma_r^2 \cos\Omega v \; e^{-\alpha(t-v)} dv$$

$$\doteq \overline{\varepsilon_{qc}^2} \; \tau + \int_0^\tau \sigma_r^2 (1 - \alpha t + \alpha v) \; dv$$

$$= \overline{\varepsilon_{qc}^2} \; \tau + \sigma_r^2 [\tau + \frac{\alpha}{2}(\tau^2 - 2\tau t)] \quad , \qquad (365)$$

where the same approximations used previously are used here. The term $(\overline{\varepsilon_{qc}^2} + \sigma_r^2)\tau$ appears in the error equation of Section VII; thus only the remaining two terms contribute to additional error.

The additional error term can now be written

$$\overline{e_{r2}^2(t)} = - 2\int_0^t \sigma_r^2 \; \frac{\alpha}{2} \; (\tau^2 - 2\tau t) \; w_2(t,\tau) \; d\tau + \overline{e_{r12}^2(t)}$$

$$= \sigma_r^2 \alpha t^3 [\frac{2}{3} \, K_{21} + \frac{5}{12} \, K_{22} t + \frac{3}{10} \, K_{23} t^2] + e_{r12}^2(t) \quad . \qquad (366)$$

The additional error in estimating $\varepsilon_p(t)\cos\eta - \varepsilon_c(t) \sin\eta$ is also given by equation 366, with Ω^2 in $\overline{e_{r12}^2(t)}$ replaced by $\Omega^2 \sin^2\eta$.

Solutions for the additional estimation error in ε_e proceeds in exactly the same manner. The additional term corresponding to equation 366 is

$$\overline{e_{r3}^2(t)} = -2 \int_o^t \sigma_r^{\,2}\alpha\Omega\left(\frac{\tau^3}{3} - \frac{t\tau^2}{2}\right) w_3(t,\tau)\, d\tau + \overline{e_{r13}^2(t)}$$

$$= \sigma_r^{\,2}\alpha\Omega t^4\left[\frac{K_{31}}{6} + \frac{7}{60} K_{32}t + \frac{4}{45} K_{33}t^2\right] + \overline{e_{r13}^2(t)} \quad . \tag{367}$$

The additional error in estimating $\varepsilon_q(t)$ from Ψ_{wb} could be found quite easily; however, in the numerical examples, this estimate is of essentially no value, so this will not be done.

XIV. APPENDIX C

This section serves to present explicit equations for the integrals of equation 280, and additional error terms due to neglecting drift rates and other components of $\underline{\Psi}$ in the decoupled mode.

The functions $f_1(t)$ and $f_2(t)$ are found from equation 127. The integrals of equation 280, after forming the integrands and using appropriate trigonometric identities, are quite straightforward; the results are given for $\omega_1 t = 2\pi$ since some simplification is obtained:

$$\int_0^t f_1{}^2(x)dx = \frac{\cos^2\eta}{4}[(1 + \frac{\Omega^2}{\omega_1{}^2})t$$

$$+\left(\frac{(1 + \frac{\Omega}{\omega_1})^2}{4(\omega_1-\Omega)} - \frac{(1 - \frac{\Omega}{\omega_1})^2}{4(\omega_1+\Omega)}\right)\sin 2\Omega t$$

$$- (1 - \frac{\Omega^2}{\omega_1{}^2}(\frac{\sin 2\Omega t}{2\Omega})] \tag{368}$$

$$\int_0^t f_2{}^2(x)dx = \frac{1}{4\omega_1{}^4}[\frac{2\Omega^2 t^3}{3} + (3 + \frac{5\Omega^2}{\omega_1{}^2})t$$

$$+ (\frac{2(1 - \frac{\Omega}{\omega_1})(\omega_1+3\Omega)}{(\omega_1+2\Omega)^2} + \frac{2(1 + \frac{\Omega}{\omega_1})}{\omega_1-2\Omega} + \frac{(1 + \frac{\Omega}{\omega_1})^2}{4(\omega_1-\Omega)}$$

$$- \frac{(1 - \frac{\Omega}{\omega_1})^2}{4(\omega_1+\Omega)} - \frac{3}{2\Omega} - \frac{\Omega}{\omega_1{}^2})\sin 2\Omega t$$

$$+ t\cos 2\Omega t(3 - (1 - \frac{\Omega}{\omega_1})(\frac{2\Omega}{\omega_1+2\Omega}) + (1 + \frac{\Omega}{\omega_1})(\frac{2\Omega}{\omega_1-2\Omega}))$$

$$+ \Omega t^2 \sin 2\Omega t] \quad , \tag{369}$$

and

$$\int_o^t f_1(x)\ f_2(x)dx = \frac{\cos\eta}{4\omega_1^{\ 2}}[\frac{2\Omega t}{\omega_1}(1 - \frac{\Omega}{\omega_1})$$

$$+ \sin 2\Omega t\left(\frac{\frac{2\Omega^2}{\omega_1}}{(\omega_1+2\Omega)^2} + \frac{(1 + \frac{\Omega}{\omega_1})}{\omega_1-2\Omega} + \frac{(1 + \frac{\Omega}{\omega_1})^2}{4(\omega_1-\Omega)}\right.$$

$$\left. - \frac{(1 - \frac{\Omega}{\omega_1})}{\omega_1+2\Omega} + \frac{(1 - \frac{\Omega}{\omega_1})^2}{4(\omega_1+\Omega)} \right) - \frac{2\Omega^2 t\ \cos 2\Omega t}{\omega_1(\omega_1+2\Omega)}] \quad . \tag{370}$$

As can be imagined, the general expressions are considerably more lengthy; thus $\omega_1 t = 2\pi$ was used in the numerical examples. The elements of the D matrix, equation 283, follow directly from equations 280, 368, 369 and 370, and will not be written out explicitly. The determinant, Δ, is obviously $D_{11}\ D_{22} - D_{12}\ D_{21}$ and will not be written explicitly for obvious reasons.

Equations 368 through 370, 288 and 289 can now be used to find errors in estimating Ψ_{od} and $\Delta V'$. Additional errors will be present due to non-zero values of ε_e, ε_q and Ψ_{eo}, as they were assumed zero in deriving the optimum filter equations. The propagation of these errors into $\Psi_e(t)$ and $\Psi_q(t)$ (in the decoupled mode) is given in equations 116 and 117. Equation 125 then gives the propagation into $\Psi_u(t)$. To evaluate these errors, ε_e and ε_q were assumed constant; errors are then given by

$$E_{A_i} = \int_o^t w_i(t,\tau)\ \Psi_{uba}(\tau)d\tau \quad , \tag{371}$$

where Ψ_{uba} represents those components of Ψ_{ub} due to these additional terms, and the subscript i indicates that errors in both $\Delta V'$ and Ψ_{od} are given by this equation. The form of $w_i(t,\tau)$ is given by equation 278; the integrals required, then, are

$$I_1 = \int_o^t f_2(\tau)\Psi_{uba}(\tau)d\tau \qquad (372)$$

and

$$I_2 = \int_o^t f_2(\tau)\Psi_{uba}(\tau)d\tau \quad . \qquad (373)$$

With the K's from equations 284 through 287, the errors can then be evaluated.

The integrals I_1 and I_2 are extremely lengthy in form, although straightforward. They are therefore given for t = 2 hours and $\omega_1 t = 2\pi$, the values used in the numerical examples of Section VIII:

$$I_1 = \cos\eta(-22\varepsilon_q - 700\varepsilon_e - 0.11\Psi_{eo}) \qquad (374)$$

$$I_2 = 1210\varepsilon_q + 3.3\text{x}10^5\ \varepsilon_e + 42\Psi_{eo} \quad . \qquad (375)$$